

**WAVE PACKET ENRICHED FINITE ELEMENTS  
FOR MULTIPHYSICS WAVE PROPAGATION PROBLEMS  
IN SOLIDS**

**AMIT KUMAR**



**DEPARTMENT OF APPLIED MECHANICS  
INDIAN INSTITUTE OF TECHNOLOGY DELHI  
JULY 2020**

© Indian Institute of Technology Delhi (IITD), New Delhi, 2020

**WAVE PACKET ENRICHED FINITE ELEMENTS  
FOR MULTIPHYSICS WAVE PROPAGATION PROBLEMS  
IN SOLIDS**

by

**AMIT KUMAR**  
Department of Applied Mechanics

*Submitted*

in fulfillment of the requirements of the degree of Doctor of Philosophy

to the



**INDIAN INSTITUTE OF TECHNOLOGY DELHI**  
**JULY 2020**

*Dedicated to  
my parents*

# Certificate

This is to certify that the thesis entitled “**Wave Packet Enriched Finite Elements for Multiphysics Wave Propagation Problems in Solids**” being submitted by **Mr. Amit Kumar** to the Indian Institute of Technology Delhi, for the award of the degree of Doctor of Philosophy in Applied Mechanics is a record of original bonafide research work carried out by him under our supervision and guidance. The thesis work, in our opinion, has reached the requisite standard fulfilling the requirements for the degree of Doctor of Philosophy.

The results contained in this thesis have not been submitted in part or full to any other University or Institute for the award of any degree or diploma.

Prof. Maloy K. Singha  
Department of Applied Mechanics  
Indian Institute of Technology Delhi  
New Delhi 110016

Prof. Santosh Kapuria  
Department of Applied Mechanics  
Indian Institute of Technology Delhi  
New Delhi 110016

# Acknowledgements

Undertaking this Ph.D. work has been an exciting journey for me and it would not have been possible to do without the support and guidance that I received from many people. I am extremely fortunate to have Prof. Santosh Kapuria as my esteemed supervisor. I express my heartiest gratitude to my supervisor who, by his able guidance and meticulous supervision, set the sense of direction to this work and provided active support and kind help at every stage of the work. I am also grateful to Prof. M. K. Singha for his support and encouragement during this work.

Special thanks to the management of CSIR-Central Mechanical Engineering Research Institute (CSIR-CMERI), for allowing me to pursue my doctoral studies. I am thankful to Dr. Avik Chatterjee, HOD, Advanced Design and Analysis Group, for his encouragement and support. I would like to extend my very special thanks to my colleagues from CSIR-CMERI for their help during this work.

Chairman and members of SRC, Prof. Suhail Ahmad, Prof. S. P. Singh and Dr. S. Pradyumna have offered many valuable suggestions. I would like to extend my sincere thanks to the members of my research group, Dr. Dhanesh N., Dr. Yaqoob Yasin, Dr. Jitendra Kumar Agrahari, Mr. Bhabagrahi N. Sharma and Mr. Mayank Jain. A very special thanks to Mr. Adnan Ahmad for his immense help with the administrative tasks. I wish to express my gratitude to all the staff of the Department of Applied Mechanics.

My special regards to my teachers at different stages of education.

I wish to dedicate this thesis to my parents, Satyapal Singh and Ishwari Devi. My hard-working parents have provided me with best education and always encouraged in pursuing higher

---

education. I also thank my brother and sister for their continuous support. I owe my deepest gratitude towards my wife, Prachi Singh, for her eternal support and understanding of my goals and aspirations. Special thanks to my in-laws for their unconditional love and support.

Amit Kumar

# Abstract

The analysis of wave propagation in solids is desired in a number of applications, for example, impact response, characterization of elastic waves for nondestructive evaluation (NDE), guided waves for structural health monitoring (SHM) of thin-walled structures, tremor propagation through ground and thermal shock waves in elastic and piezoelectric components. The finite element method (FEM), which is otherwise a robust numerical tool for solving boundary value problems, has been found wanting in case of different classes of wave propagation problems due to either prohibitively large computational efforts required or errors like numerical dispersion, dissipation and Gibbs phenomenon, introduced in the solution. The various non-conventional finite element methods proposed in the literature suffer from limitations such as the lack of generality of the standard FEM, requirement of prior knowledge of the type of solution, applicability for only a specific class of problems and involving computationally expensive inverse Laplace transform for time domain solution.

In this thesis, a new wave packet enriched finite element (FE) formulation is presented, which is capable of accurately solving a wide range of wave propagation problems in one and two dimensional elastic domains under narrowband and broadband excitations. The Lagrangian interpolation functions of the standard FEM are enriched with the local element domain spatial harmonic functions which satisfy the condition of partition of unity. The nodal displacements are used as the DOFs for the conventional Lagrangian interpolation, while the enrichment functions are made to vanish at the nodes. It allows prescription of boundary conditions in the same way as in the conventional FE method. The Newmark- $\beta$  implicit time integration is employed to solve the equations of motion in time domain. The method is assessed for different classes of wave propagation problems such as the impact and high-frequency guided wave propagation in bars and plates, and surface and body wave propagation in semi-infinite solid media for which the

---

classical FE method either fails to yield accurate results or is prohibitively expensive. It is shown that the present formulation gives accurate solutions to the former and significant improvement in computational efficiency for the latter category of problems. The performance is also assessed against other special FEs such as the spectral FE and a recently proposed enriched FE with global harmonic basis functions.

Further, a coupled multiphysics enriched FE formulation based on the L–S and G–L generalized piezothermoelasticity theories is presented for accurately solving transient electroelastic, thermoelastic and electrothermoelastic wave propagation problems in 2D elastic and piezoelectric continua. The formulation considers the full thermoelectromechanical coupling consisting of the direct and converse piezoelectric effects, thermoelastic effects and pyroelectric effects and the momentum, charge and energy balance equations. The solution for impact in a piezoelectric plate is shown to alleviate the spurious undulations in both velocity and electric potential fields, which are encountered in the conventional FE solutions. For the problem of high frequency Lamb wave actuation and sensing in a thin plate bonded with piezoelectric actuator/sensor patches, the element shows significant improvement in the computational efficiency over the conventional FE. The free edge effect of steep gradients in the shear stress distribution at the actuator-plate interface is accurately captured by the proposed element using much fewer degrees of freedom than the conventional FE. For the thermoelastic and electrothermoelastic shock wave propagation problems of generalized thermoelasticity/piezothermoelasticity, the method is assessed in comparison with the Laplace transform based analytical solutions and FE solutions with dynamic remeshing available in the literature. The present solution with a static uniform mesh captures the sharp discontinuities in various fields and the wave properties of heat conduction very accurately, practically eliminating the severe drawbacks of the conventional FE solutions. The solution is then used to study the characteristics of wave propagation in thermal shock and the effect of various time constants of the generalized theories on the same.

Finally, the wave packet enriched multifield FE formulation is extended to solve axisymmetric wave propagation problems in elastic and piezoelectric solids and its performance is assessed for the problems of axisymmetric Rayleigh wave propagation in elastic semi-infinite domain and thermal shock propagation in annular elastic disc and hollow piezoelectric cylinder.

---

## सार

टोस पदार्थों में तरंग संचरण का विश्लेषण कई अनुप्रयोगों में वाञ्छित है, उदाहरण के लिए, संघात प्रतिक्रिया, अविनाशी मूल्यांकन के लिए प्रत्यास्थ तरंगों का अभिलक्षण, पतली दीवारों वाली संरचनाओं की संरचनात्मक स्वास्थ्य निगरानी (एस० एच० एम०) के लिए निर्देशित तरंगों, भूकंप संचरण तथा प्रत्यास्थ और दाबवैद्युत घटकों में तापीय प्रघाती तरंगों। परिमित अवयव विधि (एफ० ई० एम०), जो परिसीमा मान समस्याओं को हल करने के लिए एक सुदृढ़ संख्यात्मक साधन है, या तो निषेधात्मक रूप से बड़े अभिकलनात्मक प्रयासों की आवश्यकता या संख्यात्मक परिक्षेपण और अपव्यय जैसी त्रुटियों तथा गिब्स की परिघटना के कारण तरंग संचरण समस्याओं के विभिन्न वर्गों के मामलों में अपर्याप्त पायी गयी है। साहित्य में प्रस्तावित विभिन्न गैर पारम्परिक एफ० ई० विधियां कई प्रकार की कमियों, जैसे सामान्य एफ० ई० एम० जैसी व्यापकता की कमी, समाधान के प्रकार के पूर्व ज्ञान की आवश्यकता, केवल समस्याओं के एक विशेष वर्ग के लिए उपयुक्तता तथा समय क्षेत्रीय हल के लिए अभिकलनीय रूप से महंगी *इन्वर्स लाप्लास ट्रांसफार्म* के उपयोग से सीमित हैं।

इस शोध प्रबंध में एक नया तरंग पैकेट संवर्धित परिमित अवयव सूत्रीकरण प्रस्तुत किया गया है जो *नैरोबैंड* और *ब्राडबैंड* उत्तेजन के अन्तर्गत एक और दो आयामी प्रत्यास्थ माध्यमों में तरंग संचरण समस्याओं की एक विस्तृत शृंखला को सटीक रूप से हल करने में सक्षम है। पारम्परिक एफ० ई० एम० के *लग्रांजियन* अंतर्वेशन को स्थानीय अवयव क्षेत्र के स्थानिक आवर्ती फलन से संवर्धित किया है जो *पार्टीशन ऑफ यूनिटी* की शर्त को सन्तुष्ट करते हैं। *नोडल* विस्थापन पारम्परिक *लग्रांजियन* अंतर्वेशन के लिए *डिग्री ऑफ फ्रीडम* (डी० ओ० एफ०) के रूप में उपयोग किया गया है, जबकी संवर्धन फलन *नोड्स* पर शून्य कर दिये गए हैं। यह पारम्परिक एफ० ई० विधि की तरह ही सीमा शर्तों के निर्धारण की अनुमती देता है। *न्यूमार्क-बीटा* अन्तर्निहित समाकलन को गति की समीकरणों को समय क्षेत्र में हल करने के लिए प्रयुक्त किया गया है। विधि का आंकलन तरंग संचरण के विभिन्न वर्गों की समस्याओं के लिए किया गया है जैसे छड़ों और चादरों में संघाती तथा उच्च आवृत्ति वाली निर्देशित तरंगों का संचरण, अर्द्ध अनंत टोस माध्यम में सतही और आंतरिक तरंगों का संचरण जिसके लिए पारम्परिक एफ० ई० विधि सटीक परिणाम प्राप्त करने में या तो विफल है या अभिकलनात्मक रूप से महंगी है। यहां दिखाया गया है कि वर्तमान सूत्रीकरण प्रथम श्रेणी की समस्याओं के सटीक समाधान देता है तथा परवर्ती श्रेणी की समस्याओं की अभिकलनात्मक दक्षता में महत्वपूर्ण सुधार लाता है। प्रदर्शन का मूल्यांकन अन्य विशेष एफ० ई० जैसे *स्पेक्ट्रल एफ० ई०* और हाल ही में प्रस्तावित वैश्विक आवर्ती आधार फलन वाले संवर्धित एफ० ई० के सापेक्ष भी किया गया है।

आगे, द्वि-आयामी प्रत्यास्थ और दाबवैद्युत सातत्यकों में अस्थायी विद्युत-प्रत्यास्थ, तापीयप्रत्यास्थ और वैद्युततापीय-प्रत्यास्थ तरंग संचरण समस्याओं को सटीकता से हल करने के लिए एल०-एस० और जी०-एल० सामान्यीकृत *पीजोथर्मोइलास्टिसिटी* सिद्धान्तों पर आधारित एक युग्मित *मल्टीफिजिक्स* संवर्धित एफ० ई०

---

सूत्रीकरण प्रस्तुत किया गया है। सूत्रीकरण में प्रत्यक्ष और विपरीत दाबवैद्युत प्रभाव, तापीय-प्रत्यास्थ प्रभाव और तापवैद्युत प्रभाव तथा संवेग, आवेश और ऊर्जा संतुलन समीकरणों से युक्त पूर्ण तापवैद्युतयांत्रिक युग्मन का समावेश है। एक दाबवैद्युत प्लेट में संघात के समाधान में पारम्परिक एफ० ई० समाधानों में पाये जाने वाले वेग और विद्युत क्षेत्रों के मिथ्या तरंगण को कम करना दर्शाया गया है। दाबवैद्युत प्रवर्तक और संवेदक पट्टियों से अनुबद्ध पतली प्लेट में उच्च आवृत्ति लैम्ब तरंग प्रवर्तन और संवेदन की समस्या के समाधान में प्रस्तावित संवर्धित एफ० ई० पारम्परिक एफ० ई० की तुलना में अभिकलनात्मक दक्षता में महत्वपूर्ण सुधार दिखाता है। प्रवर्तक-प्लेट अन्तर्पृष्ठ पर अपरूपण प्रतिबल वितरण में तीव्र प्रवणता के मुक्त सीमा प्रभाव को पारम्परिक एफ० ई० की तुलना में बहुत कम डी० ओ० एफ० का उपयोग करके प्रस्तावित अवयव द्वारा सटीक रूप से अभिग्रहण कर लिया गया है। सामान्यीकृत *थर्मोइलास्टिसिटी पीजोथर्मोइलास्टिसिटी* की तापप्रत्यास्थ और वैद्युततापप्रत्यास्थ प्रघाती तरंग संचरण समस्याओं के लिए विधि का मूल्यांकन साहित्य में उपलब्ध *लाप्लास* रूपांतर आधारित विश्लेषणात्मक और गतिशील *रीमेशिंग* वाले एफ० ई० हलों की तुलना में किया गया है। एक स्थिर एकसमान मेश के साथ वर्तमान हल विभिन्न क्षेत्रों में तीव्र विच्छिन्नता और ऊष्मा चालन के तरंग रूपी गुणों को शुद्धता से अभिग्रहण करते हुए पारम्परिक एफ० ई० हलों की गंभीर कमियों को दूर करता है। इस समाधान का उपयोग तापीय प्रघात में तरंग संचरण की विशेषताओं और उस पर सामान्यीकृत सिद्धान्तों के विभिन्न समय स्थिरांकों के प्रभाव के अध्ययन में किया गया है।

अन्त में, तरंग पैकेट संवर्धित *मल्टीफील्ड* एफ० ई० सूत्रीकरण को प्रत्यास्थ तथा दाबवैद्युत ठोसों में अक्ष सममित तरंग संचरण समस्याओं को हल करने के लिए बढ़ाया गया है और इसके प्रदर्शन का आंकलन प्रत्यास्थ अर्द्ध-अनंत क्षेत्रों में अक्ष सममित *रेले* तरंग संचरण तथा वलयाकार प्रत्यास्थ चकती और खोखले दाबवैद्युत *सिलिंडर* में तापीय प्रघाती तरंग संचरण समस्याओं के लिए किया गया है।

# Contents

Certificate	i
Acknowledgements	ii
Abstract	iv
Contents	viii
List of Figures	xiv
List of Tables	xxii
<b>1 Introduction</b>	<b>1</b>
1.1 PREFACE . . . . .	1
1.2 LITERATURE REVIEW . . . . .	3
1.2.1 Wave propagation in elastic solids . . . . .	4
1.2.2 Electroelastic wave propagation . . . . .	11
1.2.3 Thermoelastic wave propagation . . . . .	13
1.2.4 Electrothermoelastic wave propagation . . . . .	16
1.3 OBJECTIVES . . . . .	18

1.4	ORGANIZATION OF THE THESIS . . . . .	20
<b>2</b>	<b>Enriched Finite Element for 1D Elastic Waves</b>	<b>25</b>
2.1	INTRODUCTION . . . . .	25
2.2	GOVERNING EQUATIONS . . . . .	26
2.3	ENRICHED FE FORMULATION . . . . .	27
2.3.1	Partition of unity enrichment using wave functions . . . . .	27
2.3.2	Discretization of Hamilton's principle statement . . . . .	31
2.3.3	Imposing boundary conditions . . . . .	33
2.3.4	Computation of integrals . . . . .	34
2.4	ONE DIMENSIONAL IMPACT OF AN ELASTIC BAR . . . . .	36
2.5	TONE BURST WAVE ACTUATION IN A BAR . . . . .	41
2.5.1	Effect of enrichment parameter . . . . .	43
2.5.2	Convergence with respect to mesh size and $N_F$ . . . . .	44
2.6	WAVE PROPAGATION THROUGH A NOTCHED BAR UNDER TONE BURST EXCITATION . . . . .	47
2.7	CONCLUSIONS . . . . .	49
<b>3</b>	<b>Enriched Finite Element for 2D Elastic Waves</b>	<b>52</b>
3.1	INTRODUCTION . . . . .	52
3.2	ENRICHED FE FORMULATION IN 2D DOMAIN . . . . .	53
3.3	NUMERICAL RESULTS . . . . .	58
3.3.1	Impact in a thin plate . . . . .	58

3.3.2	Lamb wave propagation in an isotropic plate under plane strain . . . . .	60
3.3.3	2D P, S and Rayleigh waves . . . . .	62
3.3.4	Convergence study for the numerical integration of element matrices . . . . .	64
3.3.5	Effect of mesh distortion . . . . .	66
3.4	CONCLUSIONS . . . . .	66
<b>4</b>	<b>Enriched Finite Element Formulation for Generalized Piezothermoelasticity</b>	<b>69</b>
4.1	INTRODUCTION . . . . .	69
4.2	HAMILTON'S PRINCIPLE FOR GENERALIZED PIEZOTHERMOELASTICITY	70
4.3	WAVE PACKET ENRICHED MULTIFIELD FE . . . . .	72
4.4	CLOSURE . . . . .	78
<b>5</b>	<b>Electroelastic Wave Propagation in Piezoelastic Media</b>	<b>80</b>
5.1	INTRODUCTION . . . . .	80
5.2	SOLUTION FOR ELECTROELASTIC PROBLEMS . . . . .	81
5.3	LAMB WAVE PROPAGATION IN PIEZOELECTRIC ACTUATOR-PLATE- SENSOR SYSTEM . . . . .	81
5.4	LAMB WAVE PROPAGATION IN A PLATE WITH DAMAGE . . . . .	86
5.5	IMPACT IN A THIN PIEZOELECTRIC PLATE . . . . .	89
5.6	SHEAR STRESS DISTRIBUTION AT ACTUATOR-HOST STRUCTURE IN- TERFACE . . . . .	92
5.7	CONCLUSIONS . . . . .	94
<b>6</b>	<b>Thermal Shock Wave Propagation in 2D Elastic Media</b>	<b>96</b>

## CONTENTS

---

6.1	INTRODUCTION . . . . .	96
6.2	SOLUTION FOR THERMOELASTIC PROBLEMS . . . . .	96
6.3	1D THERMAL SHOCK PROPAGATION . . . . .	97
6.3.1	Comparison with Laplace solution of Wang et al. (2012) . . . . .	98
6.3.2	Comparison with FE solution of Tian et al. (2006) . . . . .	99
6.4	2D THERMAL SHOCK PROPAGATION IN PLATES . . . . .	101
6.5	BEHAVIOUR OF E AND T-WAVES UNDER THERMAL SHOCK . . . . .	103
6.5.1	Thermal and elastic wave phenomenon . . . . .	105
6.5.2	Effect of time constants . . . . .	109
6.6	CONCLUSIONS . . . . .	112
<b>7</b>	<b>Thermoelectroelastic Shock Wave Propagation in 2D Piezoelectric Media</b>	<b>115</b>
7.1	INTRODUCTION . . . . .	115
7.2	SOLUTION FOR THERMOELECTROELASTIC PROBLEMS . . . . .	116
7.3	1D THERMOELECTROELASTIC SHOCK PROPAGATION . . . . .	116
7.3.1	Comparison with Laplace solution of He et al. (2002) for L-S theory . . . . .	117
7.3.2	G-L theory solution for open circuit response . . . . .	118
7.4	2D THERMOELECTROELASTIC SHOCK PROPAGATION . . . . .	122
7.4.1	Comparison with conventional FE solutions . . . . .	124
7.4.2	Influence of pyroelectric effect on generalized piezothermoelasticity response	126
7.5	CONCLUSIONS . . . . .	130

<b>8</b>	<b>Enriched Finite Element Formulation for Axisymmetric Electrothermoelastic Waves in Solids</b>	<b>133</b>
8.1	INTRODUCTION . . . . .	133
8.2	HAMILTON'S PRINCIPLE FOR GENERALIZED PIEZOTHERMOELASTICITY	134
8.3	ENRICHED MULTIFIELD AXISYMMETRIC ELEMENT . . . . .	135
8.4	AXISYMMETRIC ELASTIC WAVE PROBLEM . . . . .	139
8.5	AXISYMMETRIC THERMOELASTIC SHOCK WAVE PROBLEM . . . . .	144
8.5.1	E and T-wave propagation behaviour . . . . .	147
8.5.2	Effect of time constants . . . . .	149
8.5.3	Effect of inner radius . . . . .	151
8.6	AXISYMMETRIC THERMOELECTROELASTIC SHOCK WAVE PROBLEM	153
8.7	CONCLUSIONS . . . . .	155
<b>9</b>	<b>Conclusions</b>	<b>158</b>
9.1	CONCLUSIONS FROM THE PRESENT WORK . . . . .	158
9.2	MAJOR CONTRIBUTIONS OF THE PRESENT WORK . . . . .	163
9.3	FUTURE SCOPE OF WORK . . . . .	165
	<b>Bibliography</b>	<b>166</b>
<b>A</b>		<b>179</b>
A.1	EXACT SOLUTION OF 1D IMPACT WAVE PROBLEM . . . . .	179
A.2	EXACT SOLUTION OF 1D TONE BURST ACTUATION PROBLEM . . . . .	180

## CONTENTS

---

<b>List of Publications from the Thesis</b>	<b>182</b>
<b>Brief Biodata of the Author</b>	<b>183</b>

# List of Figures

1.1	Time and frequency domain representations of step and toneburst signals . . . . .	4
1.2	The exact and conventional FEM solutions using 1000 linear elements for velocity in an impact problem, depicting Gibbs phenomenon in FEM solution . . . . .	6
1.3	Shape functions of the 7 <sup>th</sup> order spectral element . . . . .	8
1.4	Schematic of wave propagation based SHM using piezoelectric patches . . . . .	12
2.1	Prismatic bar . . . . .	26
2.2	Plots of cosine ( $N_{ij}^c$ ) and sine ( $N_{ij}^s$ ) enrichment terms of LHBFE-1 . . . . .	29
2.3	Plots of cosine ( $N_{ij}^c$ ) and sine ( $N_{ij}^s$ ) enrichment terms of LHBFE-2 . . . . .	30
2.4	Prismatic bar and velocity field under impact . . . . .	37
2.5	Velocity profile under impact at $t = 0.5 \mu s$ . . . . .	38
2.6	Effect of enrichment parameter $\nu$ on accuracy for the impact problem . . . . .	39
2.7	Effect of mesh size ( $m$ ) on the range of $\nu$ to obtain accurate solution for impact problem . . . . .	40
2.8	Convergence with increase in mesh refinement . . . . .	41
2.9	Prismatic aluminium bar under sinusoidal tone burst excitation . . . . .	42
2.10	Velocity profile in the bar at $t = 100 \mu s$ . . . . .	43

**LIST OF FIGURES**

---

2.11	Effect of parameters $\nu$ and $m$ on accuracy . . . . .	44
2.12	Convergence with increase in mesh size for tone burst excitation . . . . .	45
2.13	Convergence with increase in number of harmonics $N_F$ (number within bracket shown in the labels denotes $N_F$ ) . . . . .	46
2.14	Notched bar . . . . .	47
2.15	Time history of velocity at location ‘s’ on the bar under 320 kHz tone burst excitation . . . . .	48
3.1	A 4-node element in physical and natural coordinate system . . . . .	54
3.2	Plane strain section of the plate subjected to impact at $x = 0$ . . . . .	58
3.3	Longitudinal variation and time history of velocity at the midsurface in a plate under impact. The parameters within the brackets denote $n_x \times n_y$ , followed by the $m_x, m_y$ for spectral FE and $N_{F_x}, N_{F_y}, \nu$ for enriched FEs . . . . .	59
3.4	Effect of $\nu_y$ on time history of velocity at $x = 0.4$ m at midsurface obtained using present enriched FE . . . . .	60
3.5	Pin force method of excitation for Lamb wave propagation . . . . .	61
3.6	Dispersion curves for guided waves for the aluminium plate of 1.6 mm thickness .	61
3.7	Strain $\varepsilon_x$ at sensor location in a plate under ten pulse tone-burst excitation . . .	63
3.8	Geometry and force for the Rayleigh wave study . . . . .	64
3.9	Resultant displacement profile at time 0.62 s showing the P, S and Rayleigh waves in half space due to point load normal to surface . . . . .	64
3.10	Time histories of displacements $u$ and $v$ at receiver location $R_0$ . . . . .	65
3.11	Effect of mesh distortion on solution accuracy for P, S and Rayleigh wave propa- gation problem . . . . .	67

4.1	A four-node quadrilateral element in physical and natural coordinate system . . .	73
5.1	Actuator-plate-sensor configuration of Lamb wave based structural health monitoring . . . . .	82
5.2	Discretization around the piezoelectric patches for fine and coarse FE meshes employed for the conventional and enriched FE solutions respectively, for the Lamb wave propagation problem in undamaged plate . . . . .	83
5.3	Sensor potential time history for the Lamb wave actuation problem in undamaged plate. . . . .	85
5.4	RMSD for different enriched and conventional FE solutions for the Lamb wave actuation problem. With $\sim 10000$ DOFs, the present enriched FE yields an RMSD of less than 3%, while the conventional FE solution registered an RMSD of 75% . . . . .	86
5.5	Plate-transducer configuration with notch type damage . . . . .	87
5.6	Mesh discretizations near the notch for (a) fine and (b) coarse meshes employed for conventional and enriched FE solutions respectively, for the Lamb wave propagation problem in damaged plate. . . . .	87
5.7	Sensor potential history for the Lamb wave actuation problem in plate with notch-type damage. The extra band between the $S_0$ and $A_0$ modes indicates presence of damage. . . . .	88
5.8	Distribution of in-plane strain ( $\varepsilon_x$ ) over the cross-sectional area of the aluminium plate with notch damage showing Lamb waves at three time instances. . . . .	89
5.9	Plane strain section of the piezoelectric plate subjected to in-plane impact . . . .	89
5.10	Electric potential distribution at top surface of the piezoelectric plate under impact on the left edge. The enriched FE parameters shown after mesh size are $N_{F_x}^m, N_{F_y}^m, N_{F_x}^\phi, N_{F_x}^\phi$ . Drop in the electric potential is seen at the velocity wavefront.	90

**LIST OF FIGURES**

---

5.11	Velocity distributions in the piezoelectric plate under impact on the left edge. The enriched FE parameters shown after mesh size are $N_{F_x}^m, N_{F_y}^m, N_{F_x}^\phi, N_{F_x}^\phi$ . . . .	91
5.12	Velocity profiles along the piezoelectric plate subjected to impact on the left edge with the top surface under open and closed circuit conditions. . . . .	92
5.13	Meshes employed for different FE solutions for the constant voltage actuation problem . . . . .	93
5.14	Shear stress distribution at interface of piezoelectric patch and adhesive for con- stant potential actuation. The vertical line denotes the width of one enriched element from left edge. See Fig. 5.13(a) for $x'$ . . . . .	94
6.1	1D thermal shock problem . . . . .	97
6.2	Longitudinal distributions of temperature, axial displacement and stress at time $t = 0.5$ for L-S theory ( $\tau_0 = 0.5$ ) . . . . .	100
6.3	Longitudinal distributions of temperature and axial displacement at time $t = 0.5$ for G-L theory ( $\tau_1 = 0.5, \tau_2 = 0.25$ ) . . . . .	101
6.4	Longitudinal distributions of temperature and axial displacement at time $t = 0.08$ for G-L theory ( $\tau_1 = \tau_2 = 0.05$ ) . . . . .	102
6.5	2D thermal shock problem in half space . . . . .	103
6.6	Temperature distribution over the 2D domain for G-L theory at different time instances ( $\tau_1 = \tau_2 = 0.05$ ) . . . . .	104
6.7	Temperature distribution along edge OC at time $t = 0.08$ for G-L theory ( $\tau_1 =$ $\tau_2 = 0.05$ ) . . . . .	105
6.8	Distributions of temperature and displacement $u$ along edge OE at time $t = 0.08$ for G-L theory ( $\tau_1 = \tau_2 = 0.05$ ) . . . . .	106
6.9	Longitudinal profiles of temperature, axial displacement and axial stress at dif- ferent time instances for L-S theory ( $\tau_0 = 0.5$ ) . . . . .	107

6.10	Longitudinal profiles of temperature, axial displacement and axial stress at different time instances for G-L theory ( $\tau_1 = 0.5, \tau_2 = 0.25$ ) . . . . .	108
6.11	Effect of $\tau_0$ on temperature, axial displacement and axial stress profiles for L-S theory at $t = 0.5$ . . . . .	110
6.12	Effect of $\tau_2$ on temperature, axial displacement and axial stress profiles for G-L theory ( $\tau_1 = 0.5, t = 0.5$ ) . . . . .	111
6.13	Effect of $\tau_1$ on temperature, axial displacement and axial stress profiles for G-L theory ( $\tau_2 = 0.25, t = 0.5$ ) . . . . .	113
7.1	Plate with approximated 1D heat propagation . . . . .	117
7.2	Longitudinal variations of temperature, axial displacement and axial stress at time $t = 0.25$ for L-S theory ( $\tau_0 = 0.1$ ) . . . . .	119
7.3	Longitudinal variations of temperature, axial displacement and axial stress at time $t = 0.25$ for L-S theory ( $\tau_0 = 0.05$ ) . . . . .	120
7.4	Longitudinal variations of temperature, axial displacement and axial stress at time $t = 0.25$ for G-L theory ( $\tau_1 = \tau_2 = 0.05$ ) . . . . .	121
7.5	Longitudinal variations of electric potential and axial electric displacement at $t = 0.25$ for G-L theory ( $\tau_1 = \tau_2 = 0.05$ ) in the open circuit condition . . . . .	122
7.6	Plate geometry for thermal shock in piezoelectric half space . . . . .	123
7.7	Temperature contour for 2D thermal shock problem obtained using G-L theory ( $\tau_1 = \tau_2 = 0.05$ ) . . . . .	124
7.8	Electric potential contour for 2D thermal shock problem obtained using G-L theory ( $\tau_1 = \tau_2 = 0.05$ ) . . . . .	124
7.9	Temperature distribution along edges OC and OB for the 2D thermal shock problem at $t = 0.12$ for G-L theory ( $\tau_1 = \tau_2 = 0.05$ ) . . . . .	125

**LIST OF FIGURES**

---

7.10 Variations of displacement  $v$  along edge OC and  $u$  along OB for the 2D thermal shock problem at  $t = 0.12$  for G-L theory ( $\tau_1 = \tau_2 = 0.05$ ) . . . . . 126

7.11 Variations of electric potential  $\phi$  and electric field  $E_y$  along edge OC for the 2D thermal shock problem at  $t = 0.12$  for G-L theory ( $\tau_1 = \tau_2 = 0.05$ ) . . . . . 127

7.12 Electric potential  $\phi$  along edge OC at time  $t = 0.12$  (a) with and (b) without pyroelectric effect . . . . . 128

7.13 Electric field  $E_y$  along edge OC at time  $t = 0.12$  (a) with and (b) without pyroelectric effect . . . . . 129

7.14 Electric potential  $\phi$  along edge OC for L-S theory using  $\tau_0 = 0.05$  (a) with and (b) without pyroelectric effect . . . . . 130

7.15 Electric potential  $\phi$  along edge OC for G-L theory using  $\tau_1 = \tau_2 = 0.05$  (a) with and (b) without pyroelectric effect . . . . . 131

8.1 A four-node quadrilateral element in physical and natural coordinate system . . . 136

8.2 Geometry and force for the Rayleigh wave study . . . . . 140

8.3 Resultant displacement distribution at time 0.62 s showing the P, S and Rayleigh waves in half space due to point load normal to surface . . . . . 141

8.4 Time histories of displacements  $u$  and  $v$  at receiver location  $P_1$  for the axisymmetric half space problem . . . . . 141

8.5 Time histories of displacements  $u$  and  $v$  at receiver location  $P_2$  for the axisymmetric half space problem . . . . . 142

8.6 Time histories of velocities  $\dot{u}$  and  $\dot{v}$  at receiver location  $P_1$  for the axisymmetric half space problem . . . . . 142

8.7 Time histories of velocities  $\dot{u}$  and  $\dot{v}$  at receiver location  $P_2$  for the axisymmetric half space problem . . . . . 143

8.8	The geometry of the annular disc . . . . .	145
8.9	L-S theory solution for thermal shock wave propagation in a thin annular disc ( $t = 0.5, \tau_0 = 0.5$ ) . . . . .	145
8.10	G-L theory solution for thermal shock wave propagation in a thin annular disc $t = 0.5, \tau_1 = \tau_2 = 0.5$ . . . . .	146
8.11	L-S theory solution for thermal shock wave propagation in a thin annular disc at different time instances . . . . .	147
8.12	G-L theory solution for thermal shock wave propagation in a thin annular disc at different time instances . . . . .	148
8.13	L-S theory solutions for thermal shock wave propagation in a thin annular disc at $t = 0.5$ for different values of $\tau_0$ . . . . .	150
8.14	G-L theory solutions for thermal shock wave propagation in a thin annular disc at $t = 0.5$ for different values of $\tau_2$ ( $\tau_1 = 0.5$ ) . . . . .	150
8.15	G-L theory solution for thermal shock wave propagation in a thin annular disc at $t = 0.5$ for different values of $\tau_1$ ( $\tau_2 = 0.5$ ) . . . . .	151
8.16	L-S theory solution for thermal shock in annular disc at $t = 0.5$ for different inner radii $r_i$ . . . . .	152
8.17	G-L theory solution for thermal shock in annular disc at $t = 0.5$ for different inner radii $r_i$ . . . . .	152
8.18	Hollow piezoelectric cylinder subjected to thermal shock . . . . .	153
8.19	Distributions of temperature, radial stress, hoop stress and radial electric field along the thickness (mid section OC) at $t = 0.12$ for the hollow piezoelectric cylinder subjected to thermal shock using L-S theory . . . . .	154

**LIST OF FIGURES**

---

8.20 Distributions of temperature, radial displacement, radial electric field and electric potential at different time instants for the hollow piezoelectric cylinder subjected to thermal shock using L-S theory . . . . . 155

8.21 Effect of pyroelectric constant on electric potential distribution along the thickness (mid section OC) at  $t = 0.12$  (L-S theory) . . . . . 155

A.1 Prismatic bar under impact . . . . . 179

A.2 Prismatic bar of length  $L$  subjected to tone burst excitation at  $x = L$  . . . . . 180

# List of Tables

2.1	Number of DOFs in LHBFE-2 required to achieve 1.5% RMSD ( $\nu = 0.3, t = 100 \mu\text{s}$ )	47
3.1	Convergence study for number of Gauss points used for numerical integration of element matrices . . . . .	66
5.1	Material properties . . . . .	82
5.2	Mesh sizes and corresponding number of DOFs . . . . .	84