

SUB-NYQUIST CO-PRIME SENSING: TOO LITTLE CANNOT BELITTLE YOU

USHAM VIVIANO DIAS



DEPARTMENT OF ELECTRICAL ENGINEERING
INDIAN INSTITUTE OF TECHNOLOGY DELHI
JULY 2020

**SUB-NYQUIST CO-PRIME SENSING:
TOO LITTLE CANNOT BELITTLE YOU**

by

USHAM VIVIANO DIAS

DEPARTMENT OF ELECTRICAL ENGINEERING

Submitted

*in fulfillment of the requirements of the degree of Doctor of Philosophy
to the*



INDIAN INSTITUTE OF TECHNOLOGY DELHI, INDIA.

JULY 2020

...to my mother Usha Dias; grandparents Mrs. Maria & Mr. Joseph Dias and Mrs. Nightingale & Mr. Jamesbond Wartika; and my aunt Angela Inala.

Also dedicated to all the departed souls especially from my paternal and maternal extended families...

We always wish to have the best of everything that life has to offer; acquire all the available knowledge, skills, wealth, etc. But what if we are unable to do so? Does it make us small or little?... No, it does not! The little that you have might be powerful enough to make a difference that even the powerful might not.

Sub-Nyquist schemes defy the popular theory for data acquisition; acquire little and yet manage to achieve what Nyquist schemes could. Hence the belief that:

‘Too Little cannot Belittle You...

...and it cannot Belittle You because You are not Little.’

CERTIFICATE

This is to certify that the thesis titled **Sub-Nyquist Co-Prime Sensing: Too Little cannot Belittle You**, submitted by **Usham Viviano Dias**, to the Indian Institute of Technology, Delhi, for the award of the degree of **Doctor of Philosophy**, is a bona fide record of the research work done by him under my supervision.

The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

Dr. Seshan Srirangarajan

Thesis Supervisor

Department of Electrical Engineering

Indian Institute of Technology Delhi

Hauz Khas, New Delhi 110016

Place: New Delhi

ACKNOWLEDGEMENTS

I would like to thank Prof. Seshan Srirangarajan for guiding and supervising the work in this thesis, and for allowing me to explore this area on sub-Nyquist co-prime sensing. I would like to acknowledge the support received through the Visvesvaraya Phd scheme from the Ministry of Electronics and Information Technology, Government of India, and the conference fund from the Indian Institute of Technology Delhi. I would like to thank Prof. Shankar Prakriya, Prof. Monika Aggarwal, Prof. Brejesh Lall, the HoDs, all the faculty, and staff of the Electrical Department, and the Indian Institute of Technology Delhi.

A thank you to my labmates, colleagues, and friends at the Indian Institute of Technology Delhi. I thank the staff and everyone at Zanskar hostel. I got to meet nice people, and I enjoyed my stay due to all my friends (undergraduates, post-graduates, and PhD). We are of different flavours but there's something that connects.

I have received a lot; all I can do is thank them. A thank you to my teachers, the sisters, and staff of Holy Rosary Convent High School where I had six years of education; and it has been an important phase of my life. I wish to express my indebtedness to my Alma Maters: Mini-land Nursery, Ponda, St. Annes' School, Margao, Holy Rosary Convent High School, Nuvem, G. V. M's SNJA Higher Secondary School, Ponda, Padre Conceicao College of Engineering, Verna, Goa, and Vishwakarma Institute of Technology, Pune. A thank you to all the staff, my teachers, and tutors.

A special thanks to all my students for directly or indirectly contributing to my understanding; and for being a source of strength.

I would like to express my gratitude to my father, sister, aunts, uncles, relatives, neighbours, well-wishers, and friends, for life itself. They were, that's why I am.

I thank the Divine Powers that control and make things happen, they have been kind to me.

Usham Viviano Dias

ABSTRACT

Signal acquisition at Nyquist rate assures high fidelity reconstruction of the analog signal from the acquired discrete samples. Acquisition of high frequency signals require high Nyquist rates. Temporal sampling is achieved with analog-to-digital converters, but it is difficult to design these converters with extremely high sampling rate. Spatial sampling is achieved with antenna arrays, but large number of antennas for acquisition may be expensive. This has led to an interest in sub-Nyquist techniques. Moreover, some applications are only interested in the second order statistics of the signal. This is where co-prime arrays and samplers play an important role, and are considered in this thesis. The work here focuses on temporal (time-domain) signals.

Most of the literature reveals that co-prime based schemes have large latency. An attempt is made to change this notion. The difference set of the prototype co-prime array is studied in detail. The closed-form expression for the weight function is derived for the entire and continuous difference range. The weight function is of importance since it governs the accuracy and latency of the estimate. In addition, the difference set of the prototype co-prime array is studied under perturbed conditions. Two estimation strategies, blind and non-blind are described. Its computational complexity is also derived. The proposed non-blind estimation technique has larger number of contributors for estimation.

The fundamentals of the co-prime sensing based correlogram method for spectral estimation are developed. Closed-form expressions for the bias window and variance are derived. A discussion on co-prime parameter selection is also considered. The closed-form expressions derived here will provide the necessary foundation to investigate the variants of correlogram with window functions from a low latency perspective. The combined set for correlogram estimation is also compared with the cross difference set, multiplicative processor, min processor, DFT filter bank, and compressive sensing.

The difference set and correlogram theory for prototype co-prime array with mul-

tiple periods is also developed. In addition, fully sub-Nyquist and partial sub-Nyquist scenarios for cross-correlation estimation are proposed. Applications such as time-delay, range, velocity, acceleration, and cross-spectrum estimation are considered. The theory of cross spectrum and window functions is also considered.

सार

नायक्वीस्ट दर पर सिग्नल अधिग्रहण, असतत नमूनों से एनालॉग के उच्च निष्ठा पुनर्निर्माण का आश्वासन देता है। उच्च आवृत्ति संकेतों के अधिग्रहण के लिए उच्च नायक्वीस्ट दरों की आवश्यकता होती है। टेम्पोरल सैंपलिंग को एनालॉग-टू-डिजिटल कन्वर्टर्स के साथ हासिल किया जाता है, लेकिन अत्यंत उच्च नमूना दर के साथ इन कन्वर्टर्स को डिजाइन करना मुश्किल है। स्थानिक नमूने एंटेना सरणियों के साथ प्राप्त किए जाते हैं, लेकिन अधिग्रहण के लिए बड़ी संख्या में एंटेना महंगा हो सकता है। इसके कारण सब-नायक्वीस्ट तकनीकों में रुचि पैदा हुई है। इसके अलावा, कुछ एप्लिकेशन केवल सिग्नल के सेकंड आर्डर स्टैटिस्टिक्स में रुचि रखते हैं। यह वह जगह है जहाँ को-प्राइम अर्रैस और संपलेरस एक महत्वपूर्ण भूमिका निभाते हैं, और इस थीसिस में जांच की जाती है। यहां काम टेम्पोरल (टाइम-डोमेन) सिग्नल पर केंद्रित होगा।

अधिकांश साहित्य से पता चलता है कि को-प्राइम आधारित योजनाओं में बड़ी विलंबता है। इस धारणा को बदलने का प्रयास किया जाता है। प्रोटोटाइप को-प्राइम अर्रैस के डिफरेंस सेट का विस्तार से अध्ययन किया जाता है। वेट फंक्शन के लिए बंद-प्रपत्र अभिव्यक्ति संपूर्ण और निरंतर डिफरेंस रेंज के लिए व्युत्पन्न है। वेट फंक्शन का महत्व है क्योंकि यह अनुमान की सटीकता और विलंबता को नियंत्रित करता है। इसके अलावा, प्रोटोटाइप को-प्राइम अर्रैस के डिफरेंस सेट का अध्ययन विकृत स्थितियों के तहत किया जाता है। अंधे और गैर-अंधे, दो अनुमान रणनीतियों का वर्णन किया गया है। इसकी कम्प्यूटेशनल जटिलता भी व्युत्पन्न है। प्रस्तावित गैर-अंधा अनुमान तकनीक में अनुमान लगाने के लिए बड़ी संख्या में योगदानकर्ता होते हैं।

वर्णक्रमीय आकलन के लिए को-प्राइम सेंसिंग आधारित कोरलोग्राम विधि के मूल सिद्धांतों को विकसित किया गया है। बायस विंडो और वरियन्स के लिए बंद-प्रपत्र अभिव्यक्ति व्युत्पन्न हैं। को-प्राइम पैरामीटर चयन पर एक चर्चा भी किया जाता है। यहां व्युत्पन्न बंद-प्रपत्र अभिव्यक्तियाँ जांच के लिए आवश्यक आधार प्रदान करेंगी कम विलंबता के दृष्टिकोण से विंडो फंक्शंस के साथ कोरलोग्राम के वेरिएंट। यहां प्राप्त बंद-प्रपत्र अभिव्यक्ति कम विलंबता के दृष्टिकोण से विंडो फंक्शन के साथ कोरलोग्राम के वेरिएंट

की जांच करने के लिए आवश्यक आधार प्रदान करेगी। कोरेलोग्राम अनुमान के लिए संयुक्त सेट की तुलना क्रॉस-डिफरेंस सेट, गुणक प्रोसेसर, मिन प्रोसेसर, डीएफटी फिल्टर बैंक और कंप्रेसिव सेंसिंग के साथ भी की जाती है।

कई अवधियों के साथ प्रोटोटाइप को-प्राइम अर्सेस के लिए डिफरेंस सेट और कोरलोग्राम सिद्धांत भी विकसित किया गया है। इसके अलावा, क्रॉस कोरीलेशन अनुमान के लिए पूरी तरह से सब-नायक्वीस्ट और आंशिक सब-नायक्वीस्ट परिदृश्य प्रस्तावित किया हैं। समय-विलंब, सीमा, वेग, त्वरण, और क्रॉस-स्पेक्ट्रम आकलन जैसे अनुप्रयोगों पर जांच की जाती है। क्रॉस स्पेक्ट्रम और विंडो फ़ंक्शन के सिद्धांत पर भी जांच की जाती है।

TABLE OF CONTENTS

ACKNOWLEDGEMENTS	iv
ABSTRACT	v
LIST OF FIGURES	xii
LIST OF TABLES	xv
1 INTRODUCTION	1
2 RELATED WORK	3
2.1 Nyquist Rate Acquisition	3
2.2 Sub-Nyquist Acquisition	3
2.3 Compressive Wide-Band Spectrum Sensing	4
2.4 Nested and Co-Prime Arrays	5
2.5 Spectral Estimation	7
2.6 Sampling Jitter	8
2.7 Co-Prime Co-Array Interpolation	9
2.8 Cross-Correlation Estimation	9
3 CO-PRIME SENSING AND DIFFERENCE SET ANALYSIS	11
3.1 Prototype Co-Prime Sensing	13
3.1.1 Concept	13
3.1.2 Difference Set Analysis	14
3.1.3 Weight Function	21
3.2 Co-Prime Sampling Jitter	24
3.2.1 Structure under Perturbation	24
3.2.2 Difference Set under Perturbation	25
3.2.3 Weight Function under Perturbation	29
3.2.4 Computational Complexity	34

3.3	Co-Prime Sensing with Multiple Periods	37
3.3.1	Concept	37
3.3.2	Difference Set Analysis	38
3.3.3	Weight Function	43
3.3.4	Computational Complexity	44
3.4	Conclusion	47
4	CO-PRIME SENSING AND CORRELOGRAM SPECTRAL ESTIMATION	48
4.1	Co-Prime Sensing and Autocorrelation Estimation	49
4.2	Co-Prime Sensing and Correlogram Estimation	51
4.2.1	Bias Analysis	51
4.2.2	Variance Analysis	59
4.2.3	Simulation Results	64
4.3	Co-Prime Sensing with Multiple Periods and Correlogram Estimation	70
4.3.1	Bias Analysis	70
4.3.2	Variance Analysis	73
4.3.3	Simulation Results	73
4.4	Comparison with Other Methods	76
4.4.1	Cross Difference Set based Correlogram	76
4.4.2	Multiplicative Processor	79
4.4.3	Min Processor	81
4.4.4	Co-Prime DFT Filter Banks	82
4.4.5	Compressive Sensing	83
4.5	Modified Correlogram	85
4.6	Conclusion	86
5	CO-PRIME CROSS-CORRELATION ESTIMATION AND ITS APPLICATIONS	89
5.1	Fully Sub-Nyquist Co-Prime Cross-Correlation	90
5.2	Partial Sub-Nyquist Cross-Correlation	94
5.3	Comparison between Sub-Nyquist and Nyquist Scenarios	98
5.4	Cross Power Spectral Density Estimation	99
5.5	Time-Delay, Range, Velocity, and Acceleration Estimation	102

5.6	Theory of Cross Spectrum and Window Functions	105
5.7	Conclusion	107
6	CONCLUSION AND FUTURE SCOPE	108
	REFERENCES	110
	APPENDIX	117
A	Prototype Co-Prime Sensing: Proof of Bias of Correlogram for the Entire Range	117
A.1	Unbiased Autocorrelation	117
A.2	Biased Autocorrelation	118
B	Prototype Co-Prime Sensing: Proof of Bias of Correlogram for the Continuous Range	121
B.1	Biased Autocorrelation	121
C	Prototype Co-Prime Sensing: Proof of Covariance of Correlogram	124
C.1	Proof of Covariance of Correlogram for the Entire Difference Set . .	124
C.2	Proof of Covariance of Correlogram for the Continuous Difference Set	127
D	Co-Prime Sensing with Multiple Periods: Proof of Bias of Correlogram	130
E	Co-Prime Sensing with Multiple Periods: Proof of Covariance of Correlogram	133
	LIST OF PAPERS BASED ON THESIS	136
	BIOGRAPHY OF AUTHOR	137

LIST OF FIGURES

3.1	Prototype co-prime structure.	14
3.2	Self differences along with number of contributors per difference value indicated in $\{\cdot\}$ for $M = 4, N = 3$	15
3.3	Cross differences having MN distinct values each appearing once for $M = 4, N = 3$	16
3.4	Combined set $\mathcal{L}$ displaying difference values of the autocorrelation matrix with the number of contributors per difference value mentioned in $\{\cdot\}$ for $M = 4, N = 3$	17
3.5	Range of consecutive integers in the difference set.	19
3.6	Co-prime array structure without and with jitter.	24
3.7	Difference set in the presence of sampling jitters.	26
3.8	Difference set for jitter values given in (3.28).	27
3.9	Combined set of a blind system.	30
3.10	Jitter perturbed weight function prior to mapping for $M = 4$ and $N = 3$	32
3.11	Jitter perturbed weight function post-mapping for $M = 4$ and $N = 3$	33
3.12	Block diagram for the implementation of blind and non-blind systems.	35
3.13	Co-prime arrays with multiple periods.	37
3.14	Self difference sets for the co-prime array with multiple periods with $M=4, N=3$	40
3.15	Cross difference set, $x(Mn)x(Nm)$, for the co-prime array with multiple periods with $M=4, N=3$	41
3.16	Weight function $z_r(l)$ for the co-prime array with multiple periods with $M = 4$ and $N = 3$	45
4.1	Co-prime sampling example ($M = 4, N = 3$).	50
4.2	Window $w_u(l)$ and normalized bias window $W_u(e^{j\omega})$ for the prototype co-prime sampler with $M = 4$ and $N = 3$	53
4.3	Number of contributors $z(l)$ for window $w_b(l)$ and normalized $W_b(e^{j\omega})$ for the prototype co-prime sampler with $M = 4$ and $N = 3$	53
4.4	Behavior of the bias window for large values of (M, N)	55
4.5	Asymptotic relative amplitude as a function of the difference between the co-prime factors.	55

4.6	Nyquist and co-prime (entire range) sampling: Relation between sampling function and weight function.	59
4.7	Correlogram variance for the continuous set as a function of (M, N) with $\sigma^2 = 1$	60
4.8	Correlogram variance for the continuous set for large values of M and N	62
4.9	Spectral estimation (single peak) with extremely low snapshots: $M = 4, N = 3$	63
4.10	Spectral estimation (single peak) with higher snapshots: $M = 4, N = 3$	63
4.11	Convergence of the correlogram estimate.	65
4.12	Spectral estimation (single peak) for $M = 7, N = 6$ with ten snapshots.	66
4.13	Spectral estimation (single peak) for $M = 91, N = 2$ with ten snapshots.	66
4.14	Part of the weight function for extremely skewed (M, N) pair and bias window: $M = 91, N = 2$	67
4.15	Spectral estimation for consecutive (M, N) : Three spectral peaks with ten snapshots.	68
4.16	Spectral estimation for non-consecutive $(M, N) = (91, 2)$: Three spectral peaks with ten snapshots.	68
4.17	Peak position, magnitude and mean square error for single peak estimation using the entire difference set with K representing the number of snapshots.	69
4.18	Weight function $z_r(l)$ (same as Fig. 3.16) and correlogram bias window $W_b(e^{j\omega})$ for the co-prime array with multiple periods with $M = 4$ and $N = 3$	71
4.19	Bias window for $(M = 7, N = 6)$ and $(M = 7, N = 5)$	71
4.20	Co-prime arrays with multiple periods r and its effect on the relative amplitude between main-lobe and side-lobe peak.	72
4.21	Spectral estimation using the co-prime arrays with multiple periods: $M = 7, N = 6$	74
4.22	Spectral estimation using the co-prime arrays with multiple periods: $M = 7, N = 5$	75
4.23	Block diagram for cross difference set based spectral estimation.	77
4.24	Weights and bias window for cross difference set based correlogram.	78
4.25	Spectral estimation using cross difference set based correlogram.	79
4.26	Block diagram for multiplicative processing.	80

4.27	Block diagram for multiplicative processing: Alternative implementation.	80
4.28	Spectral estimation using multiplicative processing.	80
4.29	Block diagram for min processing.	81
4.30	Spectral estimation using min processing.	82
4.31	Spectral estimation using DFT filter bank.	83
4.32	Spectral estimation using compressive sensing.	85
4.33	Spectral estimation using window function with $M = 7$ and $N = 6$: Three spectral peaks with ten snapshots.	86
4.34	Spectral estimation using window function for the continuous range.	87
5.1	Signal $x_A(i)$ for fully sub-Nyquist and partial sub-Nyquist scenarios.	91
5.2	Signal $x_B(i)$ for the fully sub-Nyquist scenario.	91
5.3	Cross-correlation process for the fully sub-Nyquist scenario with $x_A(i)$ and $x_B(i)$ of length equal to one co-prime period.	92
5.4	Contributors per difference value or weight function (left) and cross- correlation estimate (right) for the fully sub-Nyquist scenario.	93
5.5	Signal $x_B(i)$ for the partial sub-Nyquist scenario.	95
5.6	Cross-correlation process for the partial sub-Nyquist scenario with $x_A(i)$ and $x_B(i)$ of length equal to one co-prime period.	96
5.7	Contributors per difference value or weight function (left) and cross- correlation estimate (right) for the partial sub-Nyquist scenario.	97
5.8	Cross power spectral density estimation: $(M, N)=(14, 13)$	100
5.9	Multiplicative processor: Cross power spectral density estimation.	101
5.10	Cross power spectral density estimation: $(M, N)=(7, 6)$	101
5.11	Application of cross-correlation for time-delay, range, velocity and ac- celeration estimation (Constant velocity scenario).	103
5.12	Application of cross-correlation for time-delay, range, velocity and ac- celeration estimation (Random velocity scenario).	104
5.13	Fully sub-Nyquist scenario for constant velocity and acceleration with $M = 7, N = 6$: Compare it with Fig. 5.11a.	105

LIST OF TABLES

3.1 Summary of unique differences per difference set for prototype co-prime array. 18

3.2 Summary of unique differences per difference set for prototype co-prime arrays with multiple periods. 39

4.1 Comparison of co-prime correlogram sensing and compressive sensing. 85

5.1 Comparison of the proposed sub-Nyquist schemes with the traditional Nyquist scheme. 99

5.2 Comparison of auto spectrum and cross spectrum. 105