# UNITS IN FINITE GROUP ALGEBRAS


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# UNITS IN FINITE GROUP ALGEBRAS

by

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*Dedicated to*

*My Family*

# Certificate

This is to certify that the thesis entitled **“Units in Finite Group Algebras”** submitted by **Ms. Neha Makhijani** to the Indian Institute of Technology Delhi, for the award of the degree of **Doctor of Philosophy**, is a record of the original bona fide research work carried out by her under our guidance and supervision. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for award of any degree or diploma.


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*New Delhi, December 2014* *Neha Makhijani*

# Abstract

Let $FG$ be the group algebra of a group $G$ over a field $F$ and $\mathcal{U}(FG)$ be its unit group. In this thesis, we investigate the algebraic structure of the unit group of certain classes of finite group algebras. We also discuss the Wedderburn decomposition of some finite semisimple algebras.

We use the Wedderburn Malcev theorem to prove that for any field $F$ and a finite group $G$, $\mathcal{U}(FG) \cong (1 + J(FG)) \rtimes \mathcal{U}(FG/J(FG))$. Therefore, to understand the structure of the unit group of $FG$, it is required to explore the structure of the normal subgroup $1 + J(FG)$ of $\mathcal{U}(FG)$ and the Wedderburn decomposition of $FG/J(FG)$.

The unit group of $FG$ is usually elusive when $|G| = 0$ in $F$. We place special emphasis on this case and begin by studying the units in $FD_{2n}$ when $F$ is a finite field of characteristic 2 and $D_{2n}$ is the dihedral group of order $2n$. If $n \geq 3$ is odd, we establish the structure of the unit group $\mathcal{U}(FD_{2n})$ and consequently compute an expression for the structure of $\mathcal{U}(FG)$, when $G$ is either $D_{4n}$ or $Q_{4n}$, the dihedral group or the generalized quaternion group of order $4n$. For any $n \geq 3$, we give the order of the group $\mathcal{U}_*(FD_{2n})$ of $*$-unitary units in $FD_{2n}$, where $*$ is the canonical involution on $FD_{2n}$. However a complete characterization of the structure of $\mathcal{U}_*(FD_{2n})$ is provided only in case $n$ is odd. For a field $F$ of characteristic $p > 2$, we also explore the unitary units in $FD_{2p^n}$.

Under suitable conditions, the application of Ferraz's version of Witt-Berman theorem and his results in [21] on the center of $FG/J(FG)$ simplifies the computation of the decomposition of $FG/J(FG)$ to a great extent. We use the same to find the decomposition of $FA_5/J(FA_5)$ when $A_5$ is an alternating group of degree 5 and $F$ is any finite field.

For an abelian group $H$, the generalized dihedral group of $H$, denoted by $Dih(H)$, is the semidirect product of $H$ and $C_2$ with $C_2$ acting(via conjugation) on $H$ by inverting elements. If $H = C_n$, then $Dih(H)$ is the usual dihedral group $D_{2n}$. We generalize some previous results and determine the structure of the unit group of $FDih(H)$ when $F$ is a finite field of characteristic 2 and $H$ is a finite abelian group of odd order $n \geq 3$.

Let $C_n \rtimes_r C_q$ be a group with presentation $\langle\ a,\ b \mid a^n,\ b^q,\ b^{-1}ab = a^r\ \rangle$ where $r$ is a non-zero integer, $q$ is a prime, $(n, rq) = 1$ and the multiplicative order of $r$ modulo $n$ is $q$. We obtain an expression to characterize the unit group of $\mathbb{F}_{q^k}(C_n \rtimes_r C_q)$ when $q$ is the multiplicative order of $r$ modulo $d$ for every divisor $d$ of $n$, $d > 1$.

In [32], Jespers, Leal and Milies have proved that there are exactly five non-isomorphic classes of finite indecomposable groups $G$ such that $G/Z(G)$ is the Klein's 4-group. We obtain a general expression for the decomposition of $FG$ whenever $G$ is any such group and $F$ is a finite field of characteristic greater than 2.

# Contents

**Certificate** **i**

**Acknowledgements** **iii**

**Abstract** **v**

**List of Tables** **ix**

**Notations** **xi**

**1 Introduction** **1**

1.1 The Jacobson Radical . . . . . . . . . . . . . . . . . . . . . . . . 2

1.2 Group Algebra . . . . . . . . . . . . . . . . . . . . . . . . . . . 6

1.3 Units in Group Algebras . . . . . . . . . . . . . . . . . . . . . . 8

1.4 Units in Finite Abelian Group Algebras . . . . . . . . . . . . . . 10

1.5 Organization of The Thesis . . . . . . . . . . . . . . . . . . . . 13

**2 Finite Dihedral and Quaternion Group Algebras** **15**

2.1 Unit Group of $\mathbb{F}_{2^k}D_{2n}$ . . . . . . . . . . . . . . . . . . . . . . 16

2.2 Unit Group of $\mathbb{F}_{2^k}D_{4n}$ and $\mathbb{F}_{2^k}Q_{4n}$ . . . . . . . . . . . . . . . . . 22

**3 Unitary Subgroup $\mathcal{U}_*(\mathbb{F}_{p^k}D_{2n})$ of $\mathcal{U}(\mathbb{F}_{p^k}D_{2n})$** **27**

3.1 Order of $\mathcal{U}_*(\mathbb{F}_{2^k}D_{2N})$ . . . . . 28

3.2 The Structure of $\mathcal{U}_*(\mathbb{F}_{2^k}D_{2n})$ . . . . . 35

3.3 The Structure of $\mathcal{U}_*\left(\mathbb{F}_{p^k}D_{2p^n}\right)$ . . . . . 41

**4 Wedderburn Decomposition of $FG/J(FG)$ 45**

4.1 Preliminaries . . . . . 46

4.2 The Wedderburn Decomposition of $FA_5/J(FA_5)$ . . . . . 47

4.2.1 $p > 5$ . . . . . 47

4.2.2 $p = 2,\ 3$ or $5$ . . . . . 48

**5 Group Algebras of Generalized Dihedral Group 61**

5.1 The Unit Group of $\mathbb{F}_{2^k}Dih(H)$ . . . . . 62

**6 The Unit Group of $\mathbb{F}_{q^k}(C_n \rtimes_r C_q)$ 65**

6.1 The Unit Group of $\mathbb{F}_{q^k}(C_n \rtimes_r C_q)$ . . . . . 65

**7 Some Finite Semisimple Group Algebras 75**

7.1 Preliminaries . . . . . 77

7.2 Wedderburn Decomposition of $\mathbb{F}_qG_1$ and $\mathbb{F}_qG_2$ . . . . . 78

7.3 The Group Algebras $\mathbb{F}_qG_3,\ \mathbb{F}_qG_4$ and $\mathbb{F}_qG_5$ . . . . . 83

7.4 Conclusion . . . . . 86

**Bibliography 87**

**Publications 93**

**Bio-Data 95**