

MULTICOMMODITY FLOWS IN PLANAR GRAPHS

NIKHIL KUMAR



DEPARTMENT OF COMPUTER SCIENCE & ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY DELHI

AUGUST 2021

© Indian Institute of Technology Delhi (IITD), New Delhi, 2021

MULTICOMMODITY FLOWS IN PLANAR GRAPHS

by

NIKHIL KUMAR

Department of computer science & Engineering

Submitted

**in fulfilment of the requirements of the degree of Doctor of Philosophy
to the**



**INDIAN INSTITUTE OF TECHNOLOGY DELHI
AUGUST 2021**

DEDICATED TO

My Family.

Certificate

This is to certify that the thesis titled **Multicommodity Flows in Planar Graphs** being submitted by **Nikhil Kumar** for the award of **Doctor of Philosophy** in Department of Computer Science and Engineering is a record of bona fide work carried out by him under my guidance and supervision at the Department of Computer Science and Engineering, Indian Institute of Technology Delhi. The work presented in this thesis has not been submitted elsewhere, either in part or full, for the award of any other degree or diploma unless otherwise stated explicitly.



Naveen Garg

Professor

Department of Computer Science and Engg.

Indian Institute of Technology Delhi

New Delhi- 110016

Acknowledgments

I am truly indebted to Naveen Garg for giving me an opportunity to pursue research under his supervision. I started working with him for my master's thesis and I can confidently say that I have learned immensely from him during this long association. His relentless pursuit of a simple, elegant solution and the focus on building up from the very basic principles has left a long-lasting impact on me. I am thankful to him for taking care of my needs, being extremely patient and allowing me the time and space to grow.

I wish to thank Amit Kumar for his many courses on algorithms. They have helped me immensely in developing a foundation necessary for doing research. I would also like to thank Ragesh Jaiswal, Amitabha Bagchi and Sandeep Sen for teaching first courses in algorithms during my formative years and Amitabha Tripathi for initiating me into research during my undergraduate days.

I have cherished memories of the two beautiful months in Grenoble and wish to thank András Sebő for being a wonderful host. I want to thank Kurt Mehlhorn and Reiji Suda for hosting me at MPII and the University of Tokyo.

My education would not have been complete without numerous discussions with fellow scholars at the department. They have helped me develop a broader perspective on many aspects of research and life; I wish to thank Jatin Batra, Syamantak Das, Shashank Sharma and Auounon Kumar for being great colleagues and friends. Looking back, the tennis ball cricket games at the institute field were a memorable experience, and I wish to thank everyone involved. I am grateful to the staff at the CSE office, especially Rekha, Hemant and Shresth, who have been extremely cooperative and helpful.

I want to thank all my friends, inside and outside IIT, for the wonderful experiences over the years; life would not have been so much fun without their company.

Lastly, I would like to thank all my family members for their immense support and understanding; their values and constant encouragement has kept me going all these years.



Nikhil Kumar

Abstract

In this thesis, we study the classical problem of multicommodity flow and cuts. In multicommodity flow problems, we are typically given an edge capacitated graph (called the supply graph) and a number of source-sink pairs. The graph formed by joining the respective source-sink pairs is called the demand graph. The objective is to route flow between the source-sink pairs without violating the edge capacities. A flow is integral if each path carries an integer flow. A flow is half-integral if the flow carried by each path is an integer multiple of half.

In sum multicommodity flow, we seek to find a flow that maximizes the total flow routed between the source-sink pairs. A set of edges whose removal disconnects all the source-sink pairs is called a multicut. The value of a multicut is at least the value of the sum-multicommodity flow, and the ratio of the minimum multicut to the maximum (integral/half-integral) sum multicommodity flow is called the (integral/half-integral) multiflow-multicut gap.

We prove a half-integral multiflow-multicut gap of 2 when the union of the supply and the demand graph is planar. To prove this result, we make an interesting connection of multicut in such instances to a connectivity augmentation problem. We also give a sequence of instances where the half-integral multiflow-multicut gap approaches 2, showing that our result is tight. We then go on to give an algorithm for computing an integral flow of value at least half the value of the half-integral flow, which implies an integral multiflow-multicut gap of 4 for such instances.

In demand multicommodity flow problem, we are given a demand value for each source-sink pair. The objective is to find a feasible routing of all the demands if possible; otherwise, give a proof that such a routing does not exist. A necessary condition for routing the demands is as follows: the total capacity of the supply edges across any cut should be at least the total value of demand edges across it. This is known as the cut-condition. The cut-condition is also sufficient when the supply graph is a tree, outerplanar, etc., but in general, it is not. The following is a natural relaxation of the cut-condition: across every cut, the total supply is at least c times the total demand, for some $c > 1$. The minimum value of c , which implies the existence of a feasible flow, is called the flow-cut gap of the instance. The minimum c , which implies the existence of

an integral flow, is called the integral flow-cut gap of the instance.

It is known that the flow-cut gap for the series-parallel graphs is exactly 2. Chekuri et al. showed that the integral flow-cut gap for series-parallel graphs is at most 5 and conjectured that it is at most 2. We make progress on this conjecture by showing a tight integral flow-cut gap of 2 when the demand graph has a special levelled structure. These instances capture the "hard" instances used to prove a lower bound of 2 for the flow-cut gap and turn out to be quite challenging. We make an interesting connection with matching in general graphs to prove our results.

Okamura-Seymour showed that the cut-condition is sufficient when all the source-sink pair lie on one face of the planar graph. Seymour showed that if the union of the supply and the demand graph is planar, then the cut-condition is again sufficient. We consider a common generalization of the two settings: any source-sink pair lies on one of the faces of the planar graph. We prove an upper bound of 3 on the flow-cut gap when the sources and sinks on a face are contiguous. We use this result to prove a flow-cut gap of $O(\log t)$ when the sources and sinks on a face may not be contiguous, where t is the maximum number of source/sink vertices on any face. We come up with a new notion of approximating demands on a face by a convex combination of non-crossing demands to prove our result.

सार

इस थीसिस में, हम बहु-वस्तु प्रवाह और कटौती की शास्त्रीय समस्या का अध्ययन करते हैं। बहु-वस्तु प्रवाह समस्याओं में, हमें आम तौर पर एक एज कैपेसिटेटेड ग्राफ (आपूर्ति ग्राफ कहा जाता है) और कई स्रोत-सिंक जोड़े दिए जाते हैं। संबंधित स्रोत-सिंक युग्मों को मिलाने से बनने वाले ग्राफ को मांग ग्राफ कहा जाता है। इसका उद्देश्य एज कैपेसिटी का उल्लंघन किए बिना सोर्स-सिंक पेयर के बीच फ्लो को रूट करना है। एक प्रवाह अभिन्न है यदि प्रत्येक पथ में एक पूर्णांक प्रवाह होता है। एक प्रवाह आधा-अभिन्न होता है यदि प्रत्येक पथ द्वारा किया गया प्रवाह आधा का पूर्णांक गुणक हो।

संक्षेप में बहु-वस्तु प्रवाह में, हम एक ऐसा प्रवाह खोजना चाहते हैं जो स्रोत-सिंक जोड़े के बीच कुल प्रवाह को अधिकतम करता है। किनारों का एक सेट जिसका निष्कासन सभी स्रोत-सिंक जोड़े को डिस्कनेक्ट करता है उसे मल्टीकट कहा जाता है। मल्टीकट का मान कम से कम योग-मल्टीकमोडिटी फ्लो का मान होता है, और न्यूनतम मल्टीकट का अधिकतम (पूर्णांक/आधा-पूर्णांक) योग और मल्टीकमोडिटी फ्लो के अनुपात को (पूर्णांक/आधा-पूर्णांक) मल्टीफ्लो-मल्टीकट अंतराल कहा जाता है।

जब आपूर्ति और मांग ग्राफ का मिलन समतल होता है, तो हम 2 का अर्ध-अभिन्न बहुप्रवाह-मल्टीकट अंतर साबित करते हैं। इस परिणाम को साबित करने के लिए, हम ऐसे मामलों में कनेक्टिविटी वृद्धि समस्या के लिए मल्टीकट का एक दिलचस्प कनेक्शन बनाते हैं। हम ऐसे उदाहरणों का एक क्रम भी देते हैं जहां अर्ध-अभिन्न बहुप्रवाह-मल्टीकट अंतर 2 के करीब पहुंचता है, यह दर्शाता है कि हमारा परिणाम तंग है। फिर हम आधे-अभिन्न प्रवाह के मूल्य के कम से कम आधे मूल्य के एक अभिन्न प्रवाह की गणना के लिए एक एल्गोरिथ्म देते हैं, जिसका अर्थ है कि ऐसे उदाहरणों के लिए 4 का एक अभिन्न बहु-प्रवाह-मल्टीकट अंतर।

मांग बहु-वस्तु प्रवाह समस्या में, हमें प्रत्येक स्रोत-सिंक जोड़ी के लिए एक मांग मूल्य दिया जाता है। इसका उद्देश्य यदि संभव हो तो सभी मांगों का एक व्यवहार्य मार्ग खोजना है; अन्यथा, इस बात का प्रमाण दें कि ऐसी रूटिंग मौजूद नहीं है। मांगों को पूरा करने के लिए एक आवश्यक शर्त इस प्रकार है: किसी भी कट में आपूर्ति किनारों की कुल क्षमता कम से कम उस पर मांग किनारों का कुल मूल्य होना चाहिए। इसे कट-कंडीशन के रूप में जाना जाता है। कटौती की स्थिति भी पर्याप्त है जब आपूर्ति ग्राफ एक पेड़, बाहरी तल, आदि है, लेकिन सामान्य तौर पर, ऐसा नहीं है। निम्नलिखित कट-शर्तों में एक स्वाभाविक छूट है: प्रत्येक कटौती के दौरान, कुल आपूर्ति कुछ $c > 1$ के लिए कुल मांग का कम से कम c गुना है। c का न्यूनतम मान, जो एक व्यवहार्य प्रवाह के अस्तित्व को दर्शाता है, इंस्टेंस का फ्लो-कट गैप कहलाता है। न्यूनतम c , जो एक अभिन्न प्रवाह के अस्तित्व को दर्शाता है, इंस्टेंस का इंटीग्रल फ्लो-कट गैप कहलाता है।

यह ज्ञात है कि श्रृंखला-समानांतर ग्राफ के लिए फ्लो-कट गैप बिल्कुल 2 है। चेकुरी एट अल। ने दिखाया कि श्रृंखला-समानांतर ग्राफ के लिए इंटीग्रल फ्लो-कट गैप अधिकतम 5 है और यह अनुमान लगाया गया है कि यह अधिकतम 2 है। हम इस अनुमान पर प्रगति करते हैं, जब डिमांड ग्राफ में एक विशेष 2 का एक तंग इंटीग्रल फ्लो-कट गैप होता है। समतल संरचना। ये उदाहरण फ्लो-कट गैप के लिए 2 की निचली सीमा साबित करने के लिए

उपयोग किए जाने वाले "कठिन" उदाहरणों को पकड़ते हैं और काफी चुनौतीपूर्ण हो जाते हैं। हम अपने परिणामों को साबित करने के लिए सामान्य ग्राफ़ में मिलान के साथ एक दिलचस्प संबंध बनाते हैं।

ओकामुरा-सीमोर ने दिखाया कि कट-कंडीशन पर्याप्त है जब सभी स्रोत-सिंक जोड़ी प्लानर ग्राफ के एक चेहरे पर है। सीमोर ने दिखाया कि यदि आपूर्ति और मांग ग्राफ का मिलन समतल है, तो कट-कंडीशन फिर से पर्याप्त है। हम दो सेटिंग्स के एक सामान्य सामान्यीकरण पर विचार करते हैं: कोई भी स्रोत-सिंक जोड़ी प्लानर ग्राफ के किसी एक चेहरे पर स्थित होती है। हम फ्लो-कट गैप पर 3 की ऊपरी सीमा साबित करते हैं जब एक चेहरे पर स्रोत और सिंक सन्निहित होते हैं। हम इस परिणाम का उपयोग $O(\log t)$ के फ्लो-कट गैप को साबित करने के लिए करते हैं, जब किसी चेहरे पर स्रोत और सिंक सन्निहित नहीं हो सकते हैं, जहां t किसी भी चेहरे पर स्रोत/सिंक शिखर की अधिकतम संख्या है। हम अपने परिणाम को साबित करने के लिए गैर-क्रॉसिंग मांगों के उत्तल संयोजन द्वारा चेहरे पर अनुमानित मांगों की एक नई धारणा के साथ आते हैं।

Contents

Certificate	ii
Acknowledgments	iii
Abstract	iv
1 Introduction	1
1.1 Multicommodity Flows and Cuts	1
1.1.1 Demand Multicommodity Flow and Sparsest Cut	2
1.1.2 Sum Multicommodity Flow and Multicut	8
1.1.3 Integral Flows and Edge Disjoint Paths	10
1.2 Our Contributions and Organization of the Thesis	13
2 Multicommodity Flows in Seymour Instances	17
2.1 Introduction	17
2.2 Preliminaries	21
2.3 Multi-Cut and 2-Edge Connectivity Augmentation	22
2.3.1 The WGMV algorithm	23
2.3.2 Interpretation of duals as flows	25
2.4 Converting Fractional Flow to Half-Integral Flow	27
2.5 Converting Half Integral Flow to Integral Flow	30
2.6 A lower bound on the half-integral flow-cut gap	32

3	Dual Half-integrality for Uncrossable Cut Cover and Maximum Half-Integral Flow	35
3.1	Introduction	35
3.2	Preliminaries	38
3.3	Half-integrality of the GW-dual for proper functions	39
3.4	The WGMV algorithm	41
3.4.1	Duals constructed by WGMV are not half-integral	42
3.5	A stronger analysis of the WGMV algorithm	43
3.6	Modifying WGMV	47
4	Multicommodity Flows in Series Parallel Graphs	53
4.1	Introduction	53
4.1.1	Results and Techniques	54
4.1.2	Related Work	55
4.2	Preliminaries	56
4.3	Routing demand in a parallel-path-graph	56
4.3.1	The Algorithm	57
4.3.2	Finding an Alternating Sequence	60
4.3.3	Alternating tree and Blossoms	62
4.3.4	Building an $s - t$ cut	67
4.4	Series-parallel graphs with levelled demands	72
5	Multicommodity Flows in Planar Graphs with Demands on Faces	75
5.1	Introduction	75
5.2	Definitions and Preliminaries	77
5.3	Dominating Demands by Laminar Families	81
5.4	Constructing Laminar Families	83
5.4.1	Dominating Laminar Families for Arbitrary Demands	87
5.5	Uncrossing Demands in Polynomial Time	88

CONTENTS	ix
5.6 Sparsest Cut	89
5.7 Discussion	90
6 Conclusions	91
Bibliography	93
List of Publications	103
Biography	105

List of Figures

- 1.1 Edges of G (supply graph) and flow paths are in blue and red respectively. All supply edges have capacity 1. There is a unit demand between (a, b) , (b, c) and (c, a) . In the figure on the left, there is a feasible (fractional) routing of all the demands, but there is no feasible integral routing. Figure on the right shows an instance with a feasible integral routing. 2
- 1.2 This example first appeared in the work of Okamura and Seymour [71]. All supply (blue) and demand (red) edges have value 1. S (green) is a cut. The total capacity of supply edges across S is three while S separates three units of demand; hence S satisfies the cut-condition. One can check that no cut violates the cut-condition. Since the source-sink of every demand is distance two apart, a total capacity of $4 \cdot 2 = 8$ is required for a feasible routing, but only six are available. Hence, no feasible routing is possible. This also implies that no more than $3/4$ of every demand can be routed simultaneously. Figure on the right shows a feasible routing of $3/4$ of every demand, which implies a flow-cut gap of $4/3$ 4
- 1.3 Supply edges are in blue and have capacity 1. (a, b) , (b, c) , (c, a) are the source-sink pairs (demand edges). Maximum (fractional) sum-multiflow is shown on the left, maximum integral sum-multiflow in the middle and minimum multicut on the right. Hence, the multiflow-multigap is $4/3$ and the integral multiflow-multicut gap is 2. 9

2.1	Edges of G are black while red edges are the source sink pairs. Capacity of all edges in G is 1. Value of maximum multiflow = $3/2$, maximum integral flow = 1, minimum multicut = 2.	18
2.2	Supply edges are black while demand edges are red. Capacity of all supply edges is 1. The dual built by WGMV is $y_{\{c\}} = y_{\{d\}} = 1/2$, $y_{\{e\}} = 3/4$ and $y_{\{b,c,d\}} = 1/4$	26
2.3	Black edges are supply while red (thick) ones are demand edges. All supply edges have capacity 1. The maximum flow value is $7/3$, while the value of the maximum half-integer (or integer) flow is 2.	27
2.4	Black edges are demand while red and green are flow paths for the respective demands. Both the flow paths carry the same amount of flow. The two flow paths are currently crossing. We reroute the flows such that the two flow paths do not cross anymore. This process is called uncrossing. There are two possible ways to reroute the flows in order to uncross, out of which exactly one is feasible: the figure on top shows the flow paths before and after a feasible rerouting, the figure on bottom shows an infeasible rerouting. For any edge, the total value of flow paths using it before and after a feasible rerouting remains the same, hence no capacities are violated because of this operation. It is easy to show that given any two crossing flow paths, there always exists a feasible rerouting to uncross them. We repeat the above operation till no crossing flow paths remain. One can show that the total "amount" of pairwise crossings decrease after each iteration of the above operation and hence the procedure terminates.	28
2.5	Gap Instance for $k = 8$. Supply edges are black while red edges are demand. Capacity of all the supply edges is 1.	32
3.1	Example showing that the duals constructed by the WGMV procedure are not half-integral	42
3.2	A is a critical set. The thick edges are the edges in H_A^i	44

4.1	An example to show how flips reduce the number of partially routed demands. The horizontal lines on the left denote the paths P_i and the vertical lines correspond to tight edges on the paths. $q^1, q^2, \dots, q^5$ are segments and $f^1, f^2, \dots, f^6$ are demands.	59
4.2	Constructing the auxiliary graph to compute an alternating sequence.	61
5.1	Gap Instance: cut condition is satisfied but no feasible flow. All supply (blue) and demand (red) edges have value 1. Since the end points of every demand edge is 2 units apart, a total of $4 \times 2 = 8$ supply edges are required for a feasible routing but only 6 are available.	76
5.2	A Face Instance	79
5.3	Creating Laminar Demands	85
5.4	C is a cut of type 4. $ \{s_l, t_{\sigma(2)}\} \cap C = 1$ but $ \{t_{\sigma(2)}, t_{\sigma(l)}\} \cap C \neq 1$. $ \{s_{k-1}, t_{\sigma(m)}\} \cap C = 1$ and $ \{t_{\sigma(m)}, t_{\sigma(k-1)}\} \cap C = 1$	86
5.5	Laminar set system S	87