SOME PROBLEMS IN INFINITE GROUP RINGS

by

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THESIS SUBMITTED TO THE INDIAN INSTITUTE OF TECHNOLOGY, DELHI
FOR THE AWARD OF THE DEGREE OF
DOCTOR OF PHILOSOPHY

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MAY 1980

ACKNOWLEDGEMENTS

I am greatly indebted to my research supervisor Dr. J.B. Srivastava for the generosity with which he gave his time for discussions, and for the invaluable guidence, constructive criticism and constant encouragement he offered during the course of my research work.

It is a pleasure to record my gratitude to my teacher Prof. P.B. Bhattacharya who created my initial interest in algebra.

I am extremely thankful to Prof. O.P. Bhutani, Head, Department of Mathematics, for his kind co-operation throughout the period of my work.

My thanks are also due to the Indian Institute of Technology, Delhi for the financial grant and other research facilities made available to me.

I avail myself this opportunity to thank my parents for their initial encouragement to undertake this work. I am also grateful to my husband Sh. Lalit Kumar and my mother-in-law Smt. Sarojben for their unstinting and spontaneous help during the entire period of work. To my daughter Ritu also I am thankful for the sunshine and joy she gave during my moments of tiredness and hardwork.

Finally, my special thanks are to Sh. V.P. Sharma for his very elegant preparation of the final typescript.

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SYNOPSIS

SOME PROBLEMS IN INFINITE GROUP RINGS

Throughout the thesis ring will mean associative ring with identity $1 \neq 0$. Let A be a ring and let G be a group. Let A[G] denote the group ring of the group G over the ring A. Thus, A[G] is the free left A-module with the elements of G forming a free basis. A typical element of A[G] is a finite formal sum $\Sigma\, a_g g$, $a_g \in A$, $g \in G$. Letting the elements of A commute with those of G, the multiplication in A[G] is induced by the group operation in G.

The algebraic study of group rings is a fairly new subject and there are a number of unsettled problems. Both the knowledge of Group Theory as well as Ring Theory is needed for any reasonable understanding of the problems and techniques in the subject. The recent book "The Algebraic structure of Group Rings" (Wiley Interscience, 1977) by D.S. Passman describes the present state of several problems in the subject of infinite group rings. This thesis was motivated by several open problems connected with the determination of the structure of certain ideals and quotients of infinite group rings. We attempt here to find necessary and sufficient conditions on the group G and the ring A so that the group ring A[G] will have certain standard ring theoretic property; among the properties considered are those of being semilocal, semiregular, weakly perfect and finite-dimensionality of all proper quotients. We give here a brief outline of the work done.

In Chapter I, we have attempted to solve the following problem.

PROBLEM : *What are all infinite groups* G *such that* K[G]/I *is finite dimensional over the field K for every non-zero ideal* I *of the group algebra* K[G] ?

This problem is intimately related to the problem of determining all just-infinite groups i.e., infinite groups with every proper quotient finite. We have obtained a complete solution to the problem if the infinite group G possesses a non-trivial normal abelian subgroup (for example solvable groups) or the FC-subgroup $\Delta(G)$ ($x \in \Delta(G)$ if and only if x has only finitely many conjugates in G) is non-trivial. We have also given two families of infinite simple groups which are solutions to the problem, and which show that every infinite group can be embedded in a group (of the same cardinality) that is a solution to the problem.

In Chapter II, we study semilocal, semiregular and weakly perfect rings and group rings. A ring R is semilocal if R/J(R) is Artinian. Here J(R) denotes the Jacobson radical. R is called semiregular if R/J(R) is regular and idempotents can be lifted modulo J(R). A ring R is left weakly perfect if and only if R satisfies the minimum condition on principal right ideals which are not direct summands. Certain type of subrings of semilocal rings are shown to be semilocal, and this has interesting applications to group rings. A characterization of semiregular rings has been obtained. Semiregular group rings have been studied in details. For nilpotent groups, we have obtained a complete characterization of semiregular group rings. Necessary and sufficient condition for the group algebra K[G] to be weakly perfect has been obtained in characteristic 0. Also in characteristic $p > 0$, necessary conditions are obtained. Some examples of interest are given.

Chapter III studies the maximal semiregular ideal $S(R)$ of a ring R. For specific examples $S(R)$ has been computed. $S(K[G])$ of the group algebras has been studied. The significance of its relation to the maximal regular ideal $M(R)$, due to McCoy, is given. Some more examples are illustrated.

Finally in Chapter IV, we have studied certain special types of subgroups of infinite group G which are very useful in determining the structure of certain important ideals in the group algebra $K[G]$. For a given ideal I of the group algebra $K[G]$, the quotient $\mathfrak{C}(I)/\Omega I$ has been shown to have typical significance, where $\mathfrak{C}(I)$ denotes the controlling subgroup of I and $\Omega I = \{x \in G \mid x-1 \in I\}$.

C O N T E N T S