A STUDY ON REGULAR GENUS AND GEM-COMPLEXITY OF 3-and 4-MANIFOLDS

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A STUDY ON REGULAR GENUS AND GEM-COMPLEXITY OF 3-and 4-MANIFOLDS

by

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Dedicated to

Shri Krishna and my parents.

Certificate

This is to certify that the thesis entitled **"A study on regular genus and gem-complexity of 3-and 4-manifolds"** submitted by **Manisha Binjola** to the Indian Institute of Technology Delhi, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other University or Institute for the award of any degree or diploma.

New Delhi
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Abstract

A *simplicial cell complex* is a collection of simplices that is closed under taking faces, and the non-empty intersection of any two simplices is a face or union of faces of both simplices. A $(d+1)$ colored graph (Γ, γ) is a graph $\Gamma = (V(\Gamma), E(\Gamma))$ with proper edge coloring $\gamma : E(\Gamma) \to \{0, 1, \ldots, d\}$. There exists a simplicial cell complex $\mathcal{K}(\Gamma)$ corresponding to each $(d+1)$ colored graph (Γ, γ) such that the simplices in $\mathcal{K}(\Gamma)$ are in one-to-one correspondence with the components of subgraphs of (Γ, γ).

Crystallization theory provides a combinatorial tool for visualising piecewise-linear (PL) manifolds using edge colored graphs. Crystallizations are a specialised class of edge colored graphs. The theory begins with the existence of crystallizations for every closed connected PL manifold, a result by Pezzana, also known as the Father of crystallization theory. The result subsequently extends to PL manifolds with boundary.

This thesis primarily deals with combinatorial invariants: regular genus and gem-complexity, and Theorems 2.3.7, 3.2.8, 3.3.3, 4.3.10, 6.3.1 and 6.3.2 give some significant results of combinatorial topology. It consists of seven chapters which comprise the preliminaries, known results, and the conclusion with future work directions. In Chapter 1, we review the preliminaries used in the subsequent chapters. In Chapter 2, we give the lower bounds for the gem-complexity of PL 3-manifolds with boundary in terms of their regular genus and number of boundary components. Further, we show the sharpness of bounds for several 3-manifolds with boundary. Moreover, we characterize the manifolds with connected boundary that attain the bounds as handlebodies i.e., the regular neighbourhood of a graph embedded in three dimensional Euclidean space. In Chapter 3, we give several lower bounds for gem-complexity and regular genus for PL 4-manifolds with boundary. These lower bounds enable strict improvements over previously known estimates for the regular genus and gem-complexity of a PL 4-manifold with boundary. Further, the sharpness of these bounds has also been shown for a large class of PL 4-manifolds with boundary. In Chapter 4, we introduce a combinatorial invariant: weighted regular genus for manifolds with boundary. We find a relation between regular genus and weighted regular genus. Further, we give a lower bound for the weighted regular genus of PL 4-manifolds with boundary. Consequently, the theorems 4.3.3 and 4.3.10 imply a lower bound for the regular genus of PL 4-manifolds with boundary. Also, the bound for the regular genus is improved over the bound obtained in Chapter 3. In Chapter 5, we study a class of closed connected orientable PL 4-manifolds admitting a semi-simple crystallization and which have an infinite cyclic fundamental group. We show that each manifold in the class admits a handle decomposition in which the number of 2-handles depends upon its second Betti number and the number of other h-handles ($h \leq 4$) is at most two. In Chapter 6, we introduce the notion of $(f_0, f_1, \dots, f_d)$-type semi-equivelar gem for closed connected d-manifolds, related to regular embedding of the gem on a surface such that the face-cycle at each vertex of the gem on the surface is of the same type. Semi-equivelar gems carry interesting geometric structures due to its symmetric properties and so make manifolds visualising easier. Given a surface S with non-negative Euler characteristic, we find all regular embedding types on S and then construct a genus-minimal semi-equivelar gem (if it exists) of each such type embedded on S. Further, we construct a semi-equivelar gem for each closed surface and for each d-manifold with regular genus at most one. In Chapter 7, we conclude the thesis and propose some future work directions.

सार

सिम्प्लिसिअल सेल कॉम्प्लेक्स, सिम्प्लिसिस का एक ऐसा समुच्चय है जो सिम्प्लिसिस के फेसज लेने पर क्लोज़्ड हो जाता है, और किन्हीं भी दो सिम्प्लिसिस का गैर-रिक्त प्रतिच्छेदन दोनों सिम्प्लिसिस का एक फेस या फेसज का संयोजन होता है। यदि ग्राफ $\Gamma(V(\Gamma),E(\Gamma))$ की एज कलरिंग $\gamma : E(\Gamma) \to \{0, 1, \ldots, d\}$ प्रॉपर होती है तो Γ को $(d+1)$-कलर्ड ग्राफ कहा जाता है। प्रत्येक $(d+1)$-कलर्ड ग्राफ (Γ, γ) के लिए एक सिम्प्लिसिअल सेल कॉम्प्लेक्स $K(\Gamma)$ मौजूद होता है, और $K(\Gamma)$ के सिम्प्लिसिस ग्राफ (Γ, γ) के एक सबग्राफ के अवयव के साथ वन-टू-वन कॉरस्पोंडेंस में होते हैं।

क्रिस्टलायजेशन थ्योरी पीसवाइज-लीनियर (PL) मेनिफोल्ड्स को एज कलर्ड ग्राफ्स के माध्यम से दृश्यीकरण करने का एक कॉम्बिनेटोरियल साधन प्रदान करता है। क्रिस्टलायजेशंस एज कलर्ड ग्राफ्स की एक विशेष श्रेणी है। पेज़ाना, जिन्हे क्रिस्टलायजेशन थ्योरी का जनक कहा जाता है, द्वारा दिए गए एक प्रमेय से थ्योरी का आरम्भ होता है जो कि प्रत्येक क्लोज़्ड कनेक्टेड PL मेनिफोल्ड के लिए एक क्रिस्टलायजेशन की उपस्तिथि को दर्शाता है। तत्पश्चात, यह परिणाम मेनिफोल्ड्स विद बाउंड्री के लिए भी सिद्ध किया गया।

यह शोध प्रबंध मुख्य रूप से कॉम्बिनेटोरियल अपरिवर्तको: रेगुलर जीनस और जेम-कॉम्प्लेक्सिटी से संबंधित है, तथा प्रमेय 2.3.7, 3.2.8, 3.3.3, 4.3.10, 6.3.1 और 6.3.2 कॉम्बिनेटोरियल टोपोलॉजी के कुछ महत्वपूर्ण परिणाम देते हैं। इसमें सात अध्याय हैं जिनमें प्रारंभिक कार्य, ज्ञात परिणाम और भविष्य के कार्य प्रस्ताव के साथ निष्कर्ष सम्मिलित हैं। अध्याय 1 में, हम आगामी अध्यायों में प्रयुक्त प्रारंभिक कार्यो की समीक्षा करते हैं। अध्याय 2 में, हम PL 3- मेनिफोल्ड्स विद बाउंड्री की जेम-कॉम्प्लेक्सिटी के निम्न परिबंध को मेनिफोल्ड के रेगुलर जीनस और बाउंड्री अवयवों की संख्या के रूप में दिखाते हैं। इसके साथ ही, हम अनेक PL 3- मेनिफोल्ड्स विद बाउंड्री के लिए परिबंध की तीक्ष्णता को दिखाते हैं। इसके अतिरिक्त, हम यह परिबंध प्राप्त करने वाले मेनिफोल्ड्स विद कनेक्टेड बाउंड्री को हैंडलबॉडीज अर्थात् 3-विमाओं के यूक्लिडियन स्पेस में अन्तर्निहित एक ग्राफ के रेगुलर नेबरहुड, के रूप में वर्णन करते हैं। अध्याय 3 में, हम PL 4-मेनिफोल्ड्स विद बाउंड्री के जेम-कॉम्प्लेक्सिटी और रेगुलर जीनस के लिए अनेक निम्न परिबंध देते हैं। ये निम्न परिबंध PL 4-मेनिफोल्ड्स विद बाउंड्री के जेम-कॉम्प्लेक्सिटी और रेगुलर जीनस के पूर्वज्ञात आंकलनो को अधिक सक्षम बनाते हैं। इसके अतिरिक्त, हमने PL 4-मेनिफोल्ड्स विद बाउंड्री के एक बड़े वर्ग के लिए इन परिबंधो की तीक्ष्णता को दिखाया है। अध्याय 4 में, हम PL 4-मेनिफोल्ड्स विद बाउंड्री के लिए एक कॉम्बिनेटोरियल अपरिवर्तक : वेटेड रेगुलर जीनस को प्रस्तुत करते हैं। हम रेगुलर जीनस और वेटेड रेगुलर जीनस के बीच एक सम्बन्ध दर्शाते हैं। इसके बाद, हम PL 4-मेनिफोल्ड्स विद बाउंड्री के वेटेड रेगुलर जीनस के लिए एक निम्न परिबंध देते हैं। परिणामतः,

प्रमेय 4.3.3 और 4.3.10, PL 4-मेनिफोल्ड्स विद बाउंड्री के रेगुलर जीनस के लिए एक निम्न परिबंध को निर्धारित करते हैं। तथा, यह निम्न परिबंध अध्याय 3 में प्राप्त किये गए रेगुलर जीनस से अधिक सक्षम है। अध्याय 5 में, हम क्लोज़्ड कनेक्टेड ओरियन्टेबल PL 4-मेनिफोल्ड्स के एक वर्ग का अध्ययन करते हैं जिनकी एक सेमि-सिंपल क्रिस्टलायजेशन है और जिनका फंडामेंटल ग्रुप इनफिनिट साइक्लिक है। हम दिखाते हैं कि इस वर्ग के प्रत्येक मेनिफोल्ड की एक ऐसी हैंडल डिकम्पोज़िशन है जिसमे 2-हैंडल्स की संख्या इसके सेकंड बैटी नंबर पर निर्भर करती है और अन्य h-हैंडल्स ($h \leq 4$) की संख्या अधिकतम 2 है। अध्याय 6 में, हम क्लोज़्ड कनेक्टेड d-मेनिफोल्ड्स के लिए (f0, f1,..., fd)-टाइप सेमि-इक्विवेलर जेम को प्रस्तुत करते हैं, जो जेम के किसी सरफेस पर एक ऐसी रेगुलर एम्बेडिंग से सम्बंधित है, जिसमें जेम के प्रत्येक वर्टेक्स की फेस साइकिल एक समान होती है। सेमि-इक्विवेलर जेम्स अपने सममित गुणों के कारण आकर्षक ज्यामितीय संरचनाएं रखते हैं और इसलिए मेनिफोल्ड्स को आसानी से दृश्यीकरण करने में सहायक होते हैं। एक सरफेस S जिसका यूलर कैरेक्टरिस्टिक गैर-ऋणात्मक है, के लिए, हम S पर सभी संभव रेगुलर एम्बेडिंग टाइप प्राप्त करते हैं, और ऐसे प्रत्येक एम्बेडिंग टाइप के लिए एक जीनस मिनिमल सेमि-इक्विवेलर जेम (यदि अस्तित्व में है) की रचना करते हैं। इसके अलावा, हम प्रत्येक क्लोज़्ड सरफेस तथा प्रत्येक मनिफोल्ड जिसका रेगुलर जीनस अधिकतम 1 है, के लिए एक सेमि-इक्विवेलर जेम की रचना करते हैं। अध्याय 7 में, हम इस शोध प्रबंध को भविष्य के कुछ प्रस्ताव कार्यो के साथ समाप्त करते हैं।

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List of Symbols

Symbol	**Meaning**
PL	Piecewise-linear,
Δ_d	Set of colors $\{0, 1, \ldots, d\}$,
$\mathbb{Z}$	Set of Integers,
$\mathbb{Z}_n$	Finite cyclic group of order n,
$V(\Gamma)$	Vertex set of graph Γ,
$\|K\|$	Geometric carrier of Simplicial cell complex K,
$K_1 \# K_2$	Connected sum of pseudomanifolds K_1 and K_2,
$\in$	belongs to,
$\chi(M)$	Euler characteristic of M,
$\cong$	isomorphic to,
$M \times N$	Direct product of manifolds M and N,
$M \mathbin{\underline{\times}} N$	Twisted product of manifolds M and N.