UNITS IN GROUP RINGS

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by

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Dedicated to
my family

Certificate

This is to certify that the thesis entitled "**UNITS IN GROUP RINGS**" submitted by "**Mr. SANDEEP MALIK**" to **Indian Institute of Technology Delhi**, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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Abstract

The unit groups of several group rings have been widely studied in the available research. In this thesis, we describe the unit groups and normalized unit groups of two integral group rings over dihedral groups. Moreover, we characterize the unit groups of some finite group algebras of all possible groups of a specific order and the groups belonging to a certain class.

This study characterizes the unit groups $\mathcal{U}(\mathbb{Z}D_8)$ and $\mathcal{U}(\mathbb{Z}D_{12})$. They can be both described as semidirect products of a group G by a cyclic group of order 2. In turn, this group G is described by using some sufficient conditions on the coefficients of the elements of the integral group ring. The normalized unit groups are also presented for both the integral group rings. Additionally, we derive some necessary conditions on a nontrivial normalized unit. Using our findings with certain results from [30, 44], we obtain torsion free normal complements of both the dihedral groups in their respective group of normalized units.

The description of $\mathcal{U}(\mathbb{F}_qG)$, where $\mathbb{F}_q$ is a finite field of order $q = p^k$ for some prime p, is quite challenging whenever $|G| = 0$ in $\mathbb{F}_q$. These instances have been duly addressed in our work. The structure of $\mathcal{U}(\mathbb{F}_{3^k}D_{6n})$ is established for a dihedral group D_{6n} of order $6n$ with n coprime to 3. Furthermore, we describe the unit group of the group algebra $\mathbb{F}_{3^k}(C_3 \times D_{2n})$. There are six non-isomorphic groups of order 42. The characterizations of the unit groups of $\mathbb{F}_qD_{42}$, $\mathbb{F}_q(C_7 \times D_6)$, $\mathbb{F}_q(C_3 \times D_{14})$ and $\mathbb{F}_qC_{42}$ are derived. Moreover, we discuss the unit groups of semisimple group algebras of remaining two groups $C_7 \rtimes C_6$ and $C_2 \times (C_7 \rtimes C_3)$ of order 42 by configuring the corresponding Wedderburn decomposition.

सार

उपलब्ध शोध में व्यापक रूप से कई ग्रुप रिंग्स के यूनिट ग्रुप्स का अध्ययन किया गया है। इस थीसिस में, हम डायहेड्रल ग्रुप्स पर दो इंटीग्रल ग्रुप रिंग्स के यूनिट ग्रुप्स और नॉर्मलाईज़ड यूनिट ग्रुप्स का अध्ययन करते हैं। इसके अलावा, हम किसी विशिष्ट आर्डर के सभी संभावित ग्रुप्स और किसी निश्चित वर्ग से सम्बंधित ग्रुप्स के कुछ परिमित ग्रुप अलजेब्राज के यूनिट ग्रुप्स की संरचना बताते हैं।

इस अध्ययन में यूनिट ग्रुप्स $\mathcal{U}(\mathbb{Z}\mathrm{D}_8)$ और $\mathcal{U}(\mathbb{Z}\mathrm{D}_{12})$ की संरचनाओं का वर्णन है। इन दोनों को आर्डर 2 के साइक्लिक ग्रुप द्वारा ग्रुप G के साथ सेमी-डायरेक्ट प्रोडक्ट के रूप में वर्णित किया जा सकता है। यहाँ इस ग्रुप G को इंटीग्रल ग्रुप रिंग के एलिमेंट्स के कोएफ़िशिएंट्स पर कुछ पर्याप्त शर्तों का उपयोग करके वर्णित किया जाता है। यहाँ दोनों इंटीग्रल ग्रुप रिंग्स के लिए नॉर्मलाईज़ड यूनिट ग्रुप्स भी प्रस्तुत किये गए हैं। इसके अतिरिक्त, हम नॉन-ट्रिविअल नॉर्मलाईज़ड यूनिट पर कुछ आवश्यक शर्तें व्युत्पन्न करते हैं। शोध पत्र [30,44] के कुछ निश्चित परिणामों के साथ अपने निष्कर्षों का उपयोग कर हमने दोनों डायहेड्रल ग्रुप्स के लिए परस्पर उनसे सम्बंधित नॉर्मलाईज़ड यूनिट ग्रुप्स के अन्दर टॉर्शन फ्री नार्मल कॉम्प्लीमेंटस प्राप्त किये हैं।

जब भी $\mathbb{F}_\mathrm{q}$ में $|\mathrm{G}| = 0$ होता है, जहाँ $\mathbb{F}_\mathrm{q}$ किसी अभाज्य संख्या p के लिए आर्डर $q = p^k$ का एक परिमित फील्ड है, यूनिट ग्रुप $\mathcal{U}(\mathbb{F}_\mathrm{q}\mathrm{G})$ का वर्णन काफी चुनौतीपूर्ण होता है। इन स्थितियों पर हमारे कार्य में विधिवत् विचार किया गया है। $\mathcal{U}(\mathbb{F}_{3^k}\mathrm{D}_{6n})$ की संरचना आर्डर $6n$ के डायहेड्रल ग्रुप D_{6n} के लिए स्थापित की गई है, जहाँ 3 संख्या n को विभाजित नहीं करता है। इसके अतिरिक्त, हमने ग्रुप अलजेब्रा $\mathbb{F}_{3^k}(\mathrm{C}_3 \times \mathrm{D}_{2n})$ के यूनिट ग्रुप का निरूपण किया है। आर्डर 42 के कुल छह नॉन- आइसोमॉर्फिक ग्रुप्स होते हैं। ग्रुप अलजेब्राज $\mathbb{F}_\mathrm{q}\mathrm{D}_{42}$, $\mathbb{F}_\mathrm{q}(\mathrm{C}_7 \times \mathrm{D}_6)$, $\mathbb{F}_\mathrm{q}(C_3 \times D_{14})$ और $\mathbb{F}_\mathrm{q}\mathrm{C}_{42}$ के यूनिट ग्रुप्स की संरचना की व्युत्पत्ति की गयी है। इसके अतिरिक्त, हमने आर्डर 42 के शेष दो ग्रुप्स $C_7 \rtimes C_6$ और $C_2 \times (C_7 \rtimes C_3)$ से सम्बंधित ग्रुप अलजेब्राज के यूनिट ग्रुप्स का वर्णन परस्पर उनकी वेडरबर्न डीकॉम्पज़िशन्स प्राप्त करके किया है।

Contents

List of Symbols

Symbol	Meaning
$\mathbb{N}$	Set of natural numbers
$\mathbb{Z}$	Set of integers
$\mathbb{Z}_n$	Set of integers upto modulo n
G	A group
$\|X\|$	Cardinality of the set X
$\widehat{X}$	$\sum\limits_{x\in X} x$, for $X \subseteq G$
$o(g)$	Order of the element g in G
$\hat{g}$	$1 + g + ... + g^{m-1}$, where $o(g) = m$
g^h	$h^{-1}gh$, the conjugate of g by h
$[g]$	Conjugacy class of g in G
$H \leq G$	H is a subgroup of G
$H \lhd G$	H is a normal subgroup of G
$[G : H]$	Index of H in G
C_n	Cyclic group of order n
$\mathbf{S}_n$	Symmetric group of degree n
$\mathbf{A}_n$	Alternating group of degree n
D_{2n}	Dihedral group of order $2n$
R	A ring with identity
$Z(R)$	Centre of R
$J(R)$	Jacobson radical of R
$\mathbf{Char}(R)$	Characteristic of R
$\mathcal{U}(R)$	Unit group of R

RG	Group ring of the group G over the ring R
$\Delta(G)$	Augmentation ideal of RG
$\mathbf{V}(RG)$	Group of normalized units of RG
$M(n, R)$	Ring of $n \times n$ matrices over R
$GL(n, R)$	General linear group of degree n over R
$\mathbb{F}$	A field
$\mathbb{F}^*$	Multiplicative group of $\mathbb{F}$
$\mathbb{F}_q$	Finite field with $\|\mathbb{F}_q\| = q = p^k$, for some prime p and a positive integer k
$\mathbb{F}_q^*$	Multiplicative group of $\mathbb{F}_q$
$a \mid b$	a divides b
$a \nmid b$	a does not divide b
$\cong$	Isomorphism
$\square$	End of a proof