

LINEAR COMBINATIONS OF LINEAR POSITIVE OPERATORS
AND GENERATING RELATIONS IN SPECIAL FUNCTIONS.

By

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CERTIFICATE

This is to certify that the thesis entitled "Linear Combinations of Linear Positive Operators and Generating Relations in Special Functions" which is being submitted by Mr. R.K.S. Rathore for the award of the degree of Doctor of Philosophy (Mathematics) to the Indian Institute of Technology, Delhi, is a record of bonafide research work. He has worked for the last two and a half years under my guidance and supervision.

The thesis has reached the standard fulfilling the requirements of the regulations relating to the degree. The results obtained in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.



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PREFACE

The linear combinations' technique is a kind of corrective surgery on linear positive operator sequences, which brings to light the inherent, but apparently disguised, rapid approximation properties of such operator sequences. The present thesis consists of a development of this theory.

We have defined a concept of local saturation in terms of the rate of convergence of approximations for certain zero-orders by a sequence of linear positive operators. Precisely, it is this saturation that controls the order of linear combinations and the approximation by them.

Also we introduce a method of exponentials which seems to work well with many of the known positive linear operator sequences. With its help the study of convergence of operators for continuous or smoother functions becomes much simplified, at the same time giving us a better grasp of the situation. It also provides us with a very powerful method for the construction of linear positive operator sequences from certain identities in **Special Functions**.

The author would be amply rewarded in his efforts if the linear combinations' theory developed here makes even a slight headway towards winning the case for the linear positive operators and also if the constructive role of special functions provides a further incentive in their study.

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CONTENTS

		<u>Page</u>
CHAPTER		
	PREFACE	
0	INTRODUCTION AND SURVEY OF CONTENTS	
	1. Preliminaries	1
	2. Survey of Contents of Chapters I-VI	5
I	ON LINEAR COMBINATIONS OF SZÁSZ-MIRAKYAN-HILLE OPERATORS AND BERNSTEIN POLYNOMIALS	
	1. Introduction	21
	2. Some basic results	23
	3. The linear combinations of Szász-Mirakyan-Hille operators	28
	4. Approximation of polynomials by $E_n^{[2k]}$	32
	5. Approximation of functions with continuous $2k$ -th derivatives	34
	6. Approximation of continuous functions	37
	7. Functions for which the approximation order given by Theorems 3 and 4 can not be improved	39
	8. Optimality of the combinations $E_n^{[2k]}$ and $L_n^{[2k]}$	47
	9. Some more results	49
	10. Best constants for Lipschitz Classes	50
II	SINGULAR INTEGRAL OF GAUSS-WEIERSTRASS AND THE SYMMETRIC MOVING AVERAGE	
	1. Introduction	56

CHAPTER		<u>Page</u>
	2. Some basic results on the Gauss-Weierstrass Integral	57
	3. Linear Combinations	59
	4. Saturation of linear combinations	62
	5. The best constants with $W_n(f;x)$	65
	6. Basic results on the symmetric Moving Average	66
	7. Application to Numerical Differentiation	71
III	LAGUERRE-BERNSTEIN SERIES OF CHENEY AND SHARMA AND OPERATORS OF MEYER-KÖNIG AND ZELLER	
	1. Introduction	78
	2. Asymptotic convergence for twice differentiable functions	81
	3. Error bound for continuous functions	84
	4. Some basic results for linear combinations	85
	5. Linear combinations	89
	6. Functions with continuous $2k$ -th derivative	90
	7. Odd derivatives	92
	8. Optimality of the linear combinations	92
	9. A generalization of the operators M_n	93
IV	A CLASS OF OPERATORS OF P. KESAVA MENON	
	1. Introduction	96
	2. Convergence, asymptotic formula and asymptotic estimates	100
	3. Basic results for linear combinations	104
	4. Linear combinations and their properties	107

<u>CHAPTER</u>		<u>Page</u>
	5. An important characterization of the class of operator sequences of Menon	111
	6. A variation diminishing property	114
	7. Generalization for functions of several variables	117
V	A GENERAL THEORY OF LINEAR COMBINATIONS	
	1. Introduction	121
	2. A general method	122
	3. Weakly regular and regular operator sequences	127
	4. Linear combinations and exponential functions	128
	5. An analogue of lethargy theorem and a uniqueness property	133
	6. A comparative enumeration of approximation of three zeroes	142
	7. An application to approximation by generalized polynomials	144
VI	SOME NEW SEQUENCES OF LINEAR POSITIVE OPERATORS FROM GENERATING RELATIONS IN SPECIAL FUNCTIONS	
	1. Introduction	151
	2. Tchebicheff polynomials of first kind	153
	3. Tchebicheff polynomials of second kind	155
	4. An identity of Carlitz	156
	5. Binomial expansion	159
	6. Hermite polynomials	161
	7. Bessel functions	164
	8. Laguerre polynomials	167

	<u>page</u>
9. Beta operators of first kind	169
10. Beta operators of second kind	171
11. Gamma operators	174
12. Sets generated by $e^t \Psi(xt)$	176
13. Another generating function for the sets $\sigma_n(x)$	178
14. A class of generating functions	181
15. Gegenbauer polynomials	183
16. Jacobi polynomials	186
17. Hypergeometric function	188
18. Legendre polynomials	190
19. Laguerre polynomials in two variables	193
20. Another generating relation	198
21. Two dimensional operators from Laguerre polynomials	200
22. Some more operators	207
REFERENCES	222