

LIE THEORETIC ORIGIN OF GENERATING FUNCTIONS

by

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SYNOPSIS

Theory of Lie groups and Lie algebra plays a fundamental role in many branches of mathematics. Research work of Lie, Engel, Cartan, Killing and Weyl have led to its recent most elegant form which is extremely useful in many applications in mathematics.

For the past three decades, mathematicians throughout the world have shown keen interest in generating functions. Various techniques have been used to derive them. Representation theory of Lie groups and Lie algebra has been applied by Weisner, Vilenkin, Miller, Kalmans and Manocha to obtain important properties and generating functions of so many special functions which arise in many branches of mathematics.

In the present work theory of Lie groups and Lie algebra have been successfully used to derive new results involving special functions. Lie groups with which we come into grips, are G_4 and $SL(2)$. Details of the contents of various chapters are as follows:

Chapter I: This chapter deals with the main theory concerning Lie groups, Lie algebra and special functions required and used throughout the thesis. It contains two parts. Part I is addressed to Lie theory and Part II to the theory of special functions.

Chapter II: Following the work of Miller (Lie theory and special functions, New York: Academic Press 1968), Weisner [Group-theoretic origins of certain generating functions, Pacific J. Math. 5, 1053-1059] ⁽¹⁹⁵⁵⁾ and Kalnins, Marocha and Miller [The Lie theory of two-variable hypergeometric functions, studies in applied mathematics, Vol. 62, No.2, April 1980].

An attempt is made to obtain generating functions associated with two-variable Horn's functions H_4 , H_7 , H_9 etc. In the entire treatment the special linear algebra $sl(2, \mathbb{C})$ plays a vital role.

Chapter III: This chapter has eight sections. Section 3.1 contains generating functions for $L_n^{(\alpha)}(x)$ by α variation. The Lie algebra associated with this work is \mathfrak{g}_4 . Section 3.2 contains generating functions for $C_n^{(\alpha)}(x)$ by α variation. Lie algebra associated with this work is $sl(2, \mathbb{C})$. Sections 3.3, 3.4 and 3.5 contain generating functions involving $P_n^{(\alpha, \beta)}(x)$ by α variation, α - β variation, α - n variation. The Lie algebra appearing in these sections is again $sl(2, \mathbb{C})$. Section 3.6 and 3.7 contain generating functions for $Q_n(c, x)$, n being a positive integer, by c -variance, c - n variation. Lie algebra associated with this work is \mathfrak{g}_4 . Finally section 3.8 contains generating functions for $Q_n(c-2n, x)$, n being a positive integer. Lie algebra associated with this section is $sl(2, \mathbb{C})$.

Chapter IV: This chapter has two sections. In section A we define a new type of function which is a special case of $P_n^{(\alpha, \beta)}(x)$ with $\beta = -\alpha$ and we name this as anti-symmetric Jacobi function. Results obtained are parallel to those obtained by Visvanathan [Generating functions for ultraspherical functions, *Canad. J. Math.* 20, 120-134 (1968)] for ultraspherical function. Section B deals with $P_n^{(\alpha-n)}(x)$, the modified ultraspherical function, in which ~~the~~ second order differential operators have been used to find generating functions.

The Lie algebra appearing in Chapter IV is $sl(2, C)$.

Chapter V: This chapter is devoted to obtaining generating functions involving the hypergeometric function ${}_2F_1[\alpha, \beta; \gamma+n; x]$. The Lie algebra appearing in this chapter is $sl(2, C)$.

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