

The Multifunction Space $L(X)$ of Cusco Maps

By

Tanvi Jain

Department of Mathematics

Submitted

*in fulfillment of the requirements
of the degree of Doctor of Philosophy*

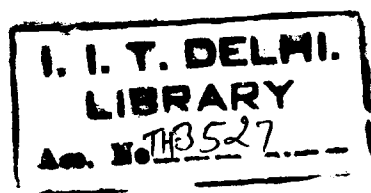
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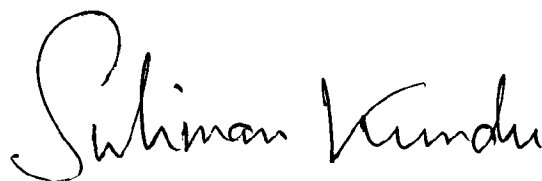


To
My Parents

Certificate

This is to certify that the thesis entitled *The Multifunction Space $L(X)$ of Cusco Maps* submitted by *Miss Tanvi Jain* to the Indian Institute of Technology Delhi, for the award of the Degree of Doctor of Philosophy, is a record of the original bona fide research work carried out by her under my guidance and supervision. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.



New Delhi
February 2008

Dr. Subiman Kundu
Supervisor

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Abstract

This thesis is devoted to the study of the multifunction space $L(X)$ of cusco maps from a Hausdorff space X to the real line $\mathbb{R}$ with the Vietoris and Fell topologies.

A set-valued mapping F from a topological space X to a topological space Y is called a cusco map if F is upper semicontinuous and $F(x)$ is a nonempty, compact and connected subset of Y for each $x \in X$. We denote by $L(X)$, the space of all subsets F of $X \times \mathbb{R}$ such that F is the graph of a cusco map from the space X to the real line $\mathbb{R}$. After introducing some basic notions on $L(X)$, we compare various well-known function space and hyperspace topologies on it. Then we study various topological properties of $L(X)$ with the Vietoris and Fell topologies in terms of the topological properties of X . More precisely, we concentrate on the study of the following topological properties of $L(X)$ with the Vietoris and Fell topologies.

- Cardinal functions such as character, pseudocharacter, tightness and weight.
- Various countability properties such as first countability, second countability and Lindelöf property.
- Metrizability, complete metrizability and submetrizability.
- Local compactness.

In addition to the topological properties listed above, we also study the separability and the countable chain condition of $L(X)$ with the Vietoris topology.

In this thesis, we also investigate how an ordered homeomorphism from $L(X)$ onto $L(Y)$, each having Vietoris topology, can be uniquely factored through a homeomorphism from X onto Y , whenever X and Y are normal.

Further, for every Hausdorff space X , the space $C(X)$ of real-valued continuous functions on X under the graph topology is a topological group that is a subspace of the space $L(X)$ under the Vietoris topology. Conditions are given showing when a positive topological group isomorphism from $C(X)$ onto $C(Y)$ has an extension to an ordered homeomorphism from $L(X)$ onto $L(Y)$. Finally the unique factorization of the ordered homeomorphisms from $L(X)$ onto $L(Y)$ is applied to obtain a unique factorization of positive topological group isomorphisms from $C(X)$ onto $C(Y)$. In fact, the existence of such an extension and the existence of such a factorization (restricted to $C(X)$) are shown to be equivalent.

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