ON MIXED FINITE ELEMENT ANALYSIS OF FOURTH ORDER ELLIPTIC SOURCE/ EIGENVALUE PROBLEMS IN CONVEX DOMAINS

by

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TO

MY FATHER

AND

TO EACH OF WHOM I OWE MORE THAN I CAN POSSIBLY EXPRESS

CERTIFICATE

This is to certify that the thesis entitled

ON MIXED FINITE ELEMENT ANALYSIS OF FOURTH ORDER ELLIPTIC SOURCE/EIGENVALUE PROBLEMS IN CONVEX DOMAINS

which is being submitted by **Ms. Neela Nataraj (n ée Neela.R)**, Research Scholar, Department of Mathematics to the INDIAN INSTITUTE OF TECHNOLOGY, DELHI for the award of the degree of Doctor of Philosophy in Mathematics, is a record of bonafide research work carried out by her under my guidance and supervision and has fulfilled all the requirements for the submission of this thesis.

The results contained in this thesis have not been submitted in part or full, to any other University or Institute for the award of any other degree or diploma.

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ABSTRACT

The present dissertation deals with the mixed finite element analysis of boundary value (source and eigenvalue) problems of fourth order elliptic linear partial differential operator Λ with variable/constant coefficients defined in convex domains $\Omega \subset \mathcal{R}^2$ with piecewise smooth curved boundary Γ, the well-known biharmonic operator and the elastic anisotropic/orthotropic/isotropic plate bending operator being the most important and interesting particular cases of this general fourth order elliptic operator. The dissertation addresses itself

1. to develop error estimates for a mixed finite element method of solution of the source and corresponding eigenvalue problems defined by fourth order elliptic operators with variable/constant coefficients in convex domains Ω with curved boundary Γ ;
2. to develop computer implementation procedures and computer codes for the mixed finite element method of solution of both source and eigenvalue problems of fourth order elliptic equations;
3. to carry out numerical experiments on a number of complex fourth order elliptic boundary value (source and eigenvalue) problems of practical interest and importance in order to establish the authenticity of the numerical results obtained and observe the performance of these methods of analysis on computers.

There are six chapters in the dissertation.

Chapter 0 contains the survey of existing research literature on mixed finite element methods of analysis of boundary value (source and eigenvalue) problems of fourth order elliptic equations with variable/constant coefficients.

Chapter 1 deals with the error estimates due to the combined effect of the approximation of the **convex curved** boundary Γ of the convex domain Ω by a **convex (straight)** polygonal boundary Γ_h and the use of numerical integration on the mixed finite element solution of the boundary value problem of the fourth order elliptic equation with variable/constant coefficients.

Error estimates for the cases:

1. when Ω is a convex polygonal domain i.e. no approximation of the boundary Γ is made,

2. when no numerical integration is done to compute the bilinear forms

are retrieved as particular cases.

Chapter 2 deals with the error estimates for the mixed finite element method of solution of the **Eigenvalue Problem** which corresponds to the **Source Problem** defined in **Chapter 1**. The **Continuous Mixed Variational Eigenvalue Problem** which will be suitable for finite element approximation is defined and the existence of its eigenpairs is proved. As in the previous chapter, the curved boundary Γ of the convex domain Ω is approximated by a straight polygonal boundary Γ_h. The corresponding **'Affine' Mixed Finite Element Eigenvalue Problem** is defined with the help of bilinear forms evaluated using suitable quadrature schemes. Using the error estimates for the corresponding source problem developed in **Chapter 1** error estimates for the eigensolutions have been obtained for the case of **simple** eigenvalues.
As in **Chapter 1**, error estimates for the cases:

1. when Ω is a convex polygonal domain i.e. **no** approximation of the boundary Γ is made.

2. when no numerical integration is done to compute the bilinear forms

are retrieved as particular cases.

In **Chapter 1**, a polygonal domain has been used to approximate the curved boundary Γ of the open convex domain Ω. In **Chapter 3**, a curved domain $\tilde{\Gamma}_h$ constructed using **Isoparametric Finite Elements** to approximate the curved boundary of Ω is used to solve the **Continuous Mixed Variational Problem** defined in **Chapter 1**. Under sufficient regularity assumptions on the exact solution of the **Continuous Mixed Variational Problem** and with proper choices of the quadrature schemes to compute the bilinear forms, error estimates for the 'isoparametric' mixed finite element approximate solution are constructed.

Chapter 4 deals with the error estimates for the (approximate) eigenvalues and corresponding eigensolutions obtained by the **Isoparametric Mixed Finite Element Method** of approximation of the **Continuous Mixed Variational Eigenvalue Problem of Chapter 2**, when the curved boundary Γ of the **convex** domain Ω is approximated by a **curved** polygonal boundary $\tilde{\Gamma}_h$

which is constructed with the help of isoparametric finite element mapping developed in **Chapter 3** for the corresponding source problem. Error Estimates for **simple** eigenvalues and the corresponding eigenfunctions have been obtained under extra-regularity assumptions on the eigensolutions.

Chapter 5 is devoted to the development of computer implementation procedures and computer codes for the mixed finite element methods of solution of the source and eigenvalue problems of fourth order elliptic operator with variable/constant coefficients defined in the convex domain Ω with boundary Γ, which have been dealt with in **Chapters 1-4**.
Numerical experiments on a good number of complex problems of practical interest and importance have been carried out following these computer implementation procedures and using these computer codes.Finally, numerical results obtained have been compared with the existing published results, if there be any, in order to establish the authenticity of the results and the efficacy of the method of analysis.

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