STRUCTURED PERTURBATION ANALYSIS OF POLYNOMIAL AND RATIONAL EIGENVALUE PROBLEMS

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by

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Dedicated to

My Family

Certificate

This is to certify that the thesis entitled **"Structured Perturbation Analysis of Polynomial and Rational Eigenvalue Problems"** submitted by **"Mr. Anshul Prajapati"** to the Indian Institute of Technology Delhi, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for award of any degree or diploma.

New Delhi
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Abstract

Nonlinear eigenvalue problems, particularly polynomial and rational eigenvalue problems, have occurred in many engineering applications and have been extensively studied in the literature. Structured backward error analysis of the computed eigenvalues/eigenpairs plays an important role in knowing the backward stability of algorithms that compute them without losing the structure of the respective eigenvalue problem. Structured eigenpair backward errors are known in the literature for matrix polynomials and rational matrix functions. Structured eigenvalue backward errors are known for matrix polynomials for some structures, but the eigenvalue backward errors of rational matrix functions are still unexplored. These backward errors have been studied for some structured matrix polynomials and structured rational matrix functions.

We derive explicit expression for optimizing the Rayleigh quotient under some constraints involving symmetric matrices in terms of the second largest eigenvalue of a parameter-dependent Hermitian matrix, which allowed us to obtain eigenvalue backward errors for higher degree matrix polynomials with skew-symmetric, T-even, T-odd, T-palindromic, and T-antipalindromic structures.

We extend the ideas of [14, 15] to obtain explicit computable formulas for the structured eigenvalue backward error of a $\lambda \in \mathbb{C}$ of rational matrix function

$$G(z) = \sum_{p=0}^{d} z^p A_p + \sum_{j=1}^{k} w_j(z) E_j,$$

where the coefficients $(A_0, A_1, \ldots, A_d, E_1, \ldots, E_k) \in \mathbb{S} \subseteq (\mathbb{C}^{n,n})^{d+k+1}$ matrices, and $w_j(z) =$

$\frac{s_j(z)}{q_j(z)}$, where $s_j(z)$ and $q_j(z)$, for $j = 1, 2, \ldots, k$ are scalar polynomials following some symmetry property. For this we reformulate the eigenvalue backward error problem into an equivalent problem of maximizing the Rayleigh quotient under Hermitian constraints or symmetric constraints. These constraints depend on the structure of the rational matrix function and the parameter d and k in $G(z)$.

Due to the relation between the eigenvalues of rational matrix functions and the associated Rosenbrock system matrix, we also consider obtaining eigenvalue backward errors for the Rosenbrock system matrix. We provide two different approaches for two different measures to compute the eigenvalue backward error of the Rosenbrock system matrix, one by reducing it into an equivalent problem of optimizing sums of two generalized Rayleigh quotients and another via the structured μ-value of a constant matrix under block diagonal perturbations.

Eigenvalue backward errors of matrix pencils/polynomial are closely related to the distance to singularity, i.e., the distance to the nearest singular matrix pencil/polynomial. This is because if $A + zE$ is any regular matrix pencil, then the distance to the nearest singular matrix pencil is bouded from below by the infimum of the eigenvalue backward error of $A + zE$ over all the scalars $\lambda \in \mathbb{C}$. Given a regular matrix pencil $A + zE$, the problem of finding the distance to singularity has attracted much work in the literature and is still an open problem. If we consider additional structure on the coefficients matrices, then the problem of computing the structured distance to singularity (distance to the nearest singular pencil with the same structure) becomes more difficult and as far as we know, this problem with structure on the pencil has not been explored in the literature. We give a purely linear algebra-based approach to derive explicit computable formulas for the distance to the nearest structured pencil $\tilde{A} + z\tilde{E}$ having a common null vector. We then obtain a family of computable lower bounds for the unstructured and structured distances to singularity. Numerical experiments suggest that, in many cases, there is a significant difference between structured and unstructured distances. We also extend this approach to structured matrix polynomials with higher degrees.

सार

नॉनलीनार आइगेनवैल्यू प्रॉब्लम्स, विशेष रूप से पोलीनोमिअल और रैशनल आइगेनवैल्यू प्रॉब्लम्स, कई इंजीनियरिंग अनुप्रयोगों में हुई हैं और साहित्य में इनका व्यापक अध्ययन किया गया है। गणना किए गए आइगेनवैल्यू / आइगेनपेयर का स्ट्रक्चर्ड बैकवर्ड एरर विश्लेषण एल्गोरिदम की बैकवर्ड स्थिरता को जानने में महत्वपूर्ण भूमिका निभाता है जो संबंधित आइगेनवैल्यू प्रॉब्लम्स की स्ट्रक्चर को खोए बिना उनकी गणना करता है। स्ट्रक्चर्ड आइगेनपेयर बैकवर्ड एरर साहित्य में मैट्रिक्स पोलीनोमिअल और रैशनल मैट्रिक्स फ़ंक्शन के लिए जानी जाती हैं। स्ट्रक्चर्ड आइगेनवैल्यू बैकवर्ड एरर कुछ स्ट्रक्चर के लिए मैट्रिक्स पोलीनोमिअल के लिए जानी जाती हैं, लेकिन रैशनल मैट्रिक्स फ़ंक्शन की आइगेनवैल्यू बैकवर्ड एरर अभी भी अज्ञात हैं। कुछ स्ट्रक्चर्ड मैट्रिक्स पोलीनोमिअल और स्ट्रक्चर्ड मैट्रिक्स फ़ंक्शन के लिए इन बैकवर्ड एरर का अध्ययन किया गया है।

हम पैरामीटर-निर्भर हर्मिटियन मैट्रिक्स के दूसरे सबसे बड़े आइगेनवैल्यू के संदर्भ में सिमेट्रिक मैट्रिक्स से जुड़े कुछ कंस्ट्रेंट्स के तहत रेले कौशैंट को ऑप्टिमाइज़ करने के लिए स्पष्ट अभिव्यक्ति प्राप्त करते हैं, जिसने हमें स्क्यू-सिमेट्रिक, T-इवन, T-ओड, T-पैलिन्ड्रोमिक, और T-एंटीपालिंड्रोमिक उच्च डिग्री मैट्रिक्स पोलीनोमिअल के लिए आइगेनवैल्यू बैकवर्ड एरर को प्राप्त करने की अनुमति दी है।

हम रैशनल मैट्रिक्स फ़ंक्शन की स्ट्रक्चर्ड आइगेनवैल्यू बैकवर्ड एरर के लिए स्पष्ट गणना योग्य सूत्र प्राप्त करने के लिए [14,15] के विचारों का विस्तार करते हैं।

$$G(z) = \sum_{p=0}^{d} z^p A_p + \sum_{j=1}^{k} w_j(z) E_j$$

जहाँ गुणांक $(A_0, A_1, \dots, A_d, E_1, \dots, E_k) \in S \subseteq (C^{n,n})^{d+k+1}$ मैट्रिसेस, और $w_j(z) = \frac{s_j(z)}{q_j(z)}$, जहाँ $s_j(z)$ और $q_j(z)$, के लिए $j = 1,2, \dots, k$ कुछ सिमेट्रि गुण का अनुसरण करने वाले स्केलर पोलीनोमिअल हैं। इसके लिए हम आइगेनवैल्यू बैकवर्ड एरर समस्या को हर्मिटियन कंस्ट्रेंट्स या सिमेट्रिक कंस्ट्रेंट्स के तहत रेले कौशेंट को अधिकतम करने की एक समतुल्य समस्या में सुधारते हैं। ये कंस्ट्रेंट्स रैशनल मैट्रिक्स फ़ंक्शन की स्ट्रक्चर और $G(z)$ में पैरामीटर d और k पर निर्भर करती हैं।

रैशनल मैट्रिक्स फ़ंक्शनों के आइगेनवैल्यू और संबंधित रोसेनब्रॉक सिस्टम मैट्रिक्स के बीच संबंध के कारण, हम रोसेनब्रॉक सिस्टम मैट्रिक्स के लिए आइगेनवैल्यू बैकवर्ड एरर प्राप्त करने पर भी विचार करते हैं। हम रोसेनब्रॉक सिस्टम मैट्रिक्स की आइगेनवैल्यू बैकवर्ड एरर की गणना करने के लिए दो अलग-अलग उपायों के लिए दो अलग-अलग दृष्टिकोण प्रदान करते हैं, एक इसे दो सामान्यीकृत रेले कौशेंट के योग को अनुकूलित करने की एक समतुल्य समस्या में रिड्यूस करके और दूसरा ब्लॉक डायगोनल परटर्बेशन के तहत एक कांस्टेंट मैट्रिक्स के स्ट्रक्चर्ड μ-वैल्यू के माध्यम से।

मैट्रिक्स पेंसिल/पोलीनोमिअल की आइगेनवैल्यू बैकवर्ड एरर डिस्टेंस टू सिंगुलैरिटी से संबंधित हैं, अर्थात, निकटतम सिंगुलर मैट्रिक्स पेंसिल/पोलीनोमिअल की दूरी। ऐसा इसलिए है क्योंकि यदि $A + sE$ कोई रेगुलर मैट्रिक्स पेंसिल है, तो निकटतम सिंगुलर मैट्रिक्स पेंसिल की दूरी सभी स्केलर $\lambda \in \mathrm{C}$ पर $A + sE$ की आइगेनवैल्यू बैकवर्ड एरर के निम्नतम द्वारा नीचे से निर्धारित की जाती है। एक रेगुलर मैट्रिक्स पेंसिल $A + sE$ दिए जाने पर, डिस्टेंस टू सिंगुलैरिटी ज्ञात करने की समस्या ने साहित्य में बहुत काम किया है और यह अभी भी एक खुली समस्या है। यदि हम कोएफ़िशिएंट मैट्रिक्स पर अतिरिक्त स्ट्रक्चर पर विचार करते हैं, तो स्ट्रक्चर्ड डिस्टेंस टू सिंगुलैरिटी (समान स्ट्रक्चर के साथ निकटतम सिंगुलर पेंसिल की दूरी) की गणना करने की समस्या अधिक कठिन हो जाती है और जहाँ तक हम जानते हैं, मैट्रिक्स पेंसिल पर स्ट्रक्चर के साथ इस समस्या का साहित्य में पता नहीं लगाया गया है। हम एक कॉमन नल्ल स्पेस वाले निकटतम स्ट्रक्चर्ड पेंसिल $\widetilde{\mathrm{A}} + \mathrm{z}\widetilde{\mathrm{E}}$ की दूरी के लिए स्पष्ट गणना योग्य सूत्र प्राप्त करने के लिए एक विशुद्ध रूप से रैखिक बीजगणित-आधारित दृष्टिकोण देते हैं। फिर हम सिंगुलैरिटी के लिए अनस्ट्रक्चर्ड और स्ट्रक्चर्ड दूरियों के लिए गणना योग्य निचली सीमाओं का एक परिवार प्राप्त करते हैं। संख्यात्मक प्रयोगों से पता चलता है कि, कई मामलों में, स्ट्रक्चर्ड और अनस्ट्रक्चर्ड दूरियों के बीच एक महत्वपूर्ण अंतर है। हम इस दृष्टिकोण को उच्च डिग्री वाले स्ट्रक्चर्ड मैट्रिक्स पोलीनोमिअल तक भी बढ़ाते हैं।

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