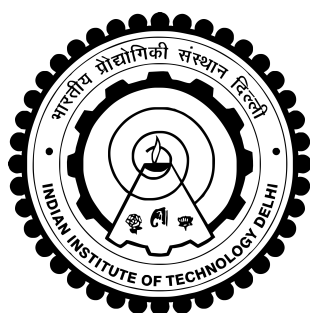


STRUCTURED PERTURBATION ANALYSIS OF POLYNOMIAL AND RATIONAL EIGENVALUE PROBLEMS

ANSHUL PRAJAPATI



DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY DELHI
JUNE 2024

STRUCTURED PERTURBATION ANALYSIS OF POLYNOMIAL AND RATIONAL EIGENVALUE PROBLEMS

by

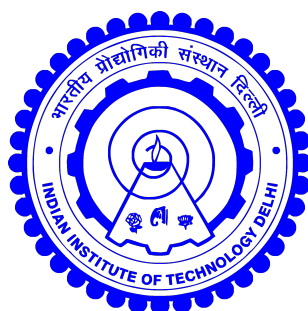
ANSHUL PRAJAPATI

Department of Mathematics

Submitted

in fulfillment of the requirements of the degree of Doctor of Philosophy

to the



INDIAN INSTITUTE OF TECHNOLOGY DELHI
JUNE 2024

Dedicated to
My Family

Certificate

This is to certify that the thesis entitled “**Structured Perturbation Analysis of Polynomial and Rational Eigenvalue Problems**” submitted by “**Mr. Anshul Prajapati**” to the Indian Institute of Technology Delhi, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for award of any degree or diploma.

New Delhi

June, 2024

Prof. Punit Sharma

Associate Professor

Department of Mathematics

Indian Institute of Technology Delhi

New Delhi 110016

Acknowledgements

This thesis marks the end of a beautiful journey to achieve my Ph.D. Several people have helped and led me along this journey. I would like to use this opportunity to express my gratitude to all those people.

First and foremost, I would like to convey my heartfelt appreciation to my supervisor, Prof. Punit Sharma, for providing me the opportunity to work with him. His guidance has been truly invaluable, and I am immensely grateful for his constant encouragement and utmost care throughout the time of research. I am very much inspired by his positive attitude, punctuality, and responsibility towards work. He is very kind and always supported me both academically and personally. I could not have thought of having a better supervisor for my Ph.D. study. I am grateful to him for introducing me to Numerical Linear Algebra and for all his contributions of time and ideas to make my Ph.D. experience productive. I am forever grateful for his guidance and mentorship.

I would like to thank Prof. Rafikul Alam and Prof. Shreemayee Bora for the many fruitful discussions we had during my visit to IIT Guwahati. I have learned a lot from their approach towards research and work ethics. I am also thankful to Balasubramaniam, Shohidul, Mandeep, and other friends at IIT Guwahati for their help and support in making my stay comfortable.

I am grateful to Prof. S. K. Neogy, who was an external expert in my upgradation seminar from JRF to SRF. He has always encouraged and supported me. I am thankful for the valuable discussions we had during the initial phase of my Ph.D. study.

I acknowledge the IIT Delhi authorities for providing all the required resources for my Ph.D. I thank my Student Research Committee members, Prof. Sivananthan Sampath, Prof. Ritumoni Sarma, and Prof. Subashish Dutta, for their important time and recommendations. Special appreciation to Prof. Aparna

Mehra, Head of the Mathematics Department, for her unwavering support. I thank all the faculty members and staff of the Department of Mathematics, IIT Delhi, for their support.

The financial support provided by IIT Delhi through RSTA and RETA to present my work at international conferences is greatly acknowledged. I am also grateful to the Ministry of Science and Technology, Government of India, for providing CSIR Ph.D. grant in the form of a monthly fellowship throughout my Ph.D. studies.

I am extremely grateful to Prof. Mukund Madhav Mishra, Hansraj College, for his encouragement and support during my graduation. I am also thankful to Mr. Sanjeev Mehndiratta, Mr. Manu Malhotra, and Mr. Vijit Sharma for their support during my initial years of education.

Thanks are due to my friends; I reached out here only because of them. Throughout my educational life, from schooling to graduation and until now, my friends have been my supporting pillars. My most significant appreciation goes to my colleagues, Abhilash, Ritika, and Sachin, for always believing in me and making my time at IIT Delhi pleasant. Other friends I must mention who have helped me directly or indirectly are: Himanshu, Kavin, Jagan, Shweta, Satyam, Aastha, Praveen, Gurdev, Sushmita, Raju, Jaya, Bhazwa, Vidyasagar, Kshitiz, Mohit, Vaibhav, Navneet, and my badminton friends Nitin, Himanshu, Jaswant, Abhilash, Sheelendra, and Gyanendra for their constant help and pleasant company. I also want to thank my friend Nidhi for keeping me motivated throughout this journey.

My acknowledgment would not be complete without expressing gratitude to my family, my parents, Roop Chand and Mamta, my brothers Ankit and Rajan, my sister-in-law Kamini, and other family members for their unconditional love, care, and belief in me. Without their support, things would have been different. Their constant encouragement, patience, and love have always kept me motivated and high. I also thank my nieces Ojasvi, Aarvi and Manvi for their love and non-stop meaningless talks.

And lastly, I thank the Almighty, who has endowed me with the perseverance and wisdom to see this study through to its successful conclusion after all these years.

Abstract

Nonlinear eigenvalue problems, particularly polynomial and rational eigenvalue problems, have occurred in many engineering applications and have been extensively studied in the literature. Structured backward error analysis of the computed eigenvalues/eigenpairs plays an important role in knowing the backward stability of algorithms that compute them without losing the structure of the respective eigenvalue problem. Structured eigenpair backward errors are known in the literature for matrix polynomials and rational matrix functions. Structured eigenvalue backward errors are known for matrix polynomials for some structures, but the eigenvalue backward errors of rational matrix functions are still unexplored. These backward errors have been studied for some structured matrix polynomials and structured rational matrix functions.

We derive explicit expression for optimizing the Rayleigh quotient under some constraints involving symmetric matrices in terms of the second largest eigenvalue of a parameter-dependent Hermitian matrix, which allowed us to obtain eigenvalue backward errors for higher degree matrix polynomials with skew-symmetric, T -even, T -odd, T -palindromic, and T -antipalindromic structures.

We extend the ideas of [14, 15] to obtain explicit computable formulas for the structured eigenvalue backward error of a $\lambda \in \mathbb{C}$ of rational matrix function

$$G(z) = \sum_{p=0}^d z^p A_p + \sum_{j=1}^k w_j(z) E_j,$$

where the coefficients $(A_0, A_1, \dots, A_d, E_1, \dots, E_k) \in \mathbb{S} \subseteq (\mathbb{C}^{n,n})^{d+k+1}$ matrices, and $w_j(z) =$

$\frac{s_j(z)}{q_j(z)}$, where $s_j(z)$ and $q_j(z)$, for $j = 1, 2, \dots, k$ are scalar polynomials following some symmetry property. For this we reformulate the eigenvalue backward error problem into an equivalent problem of maximizing the Rayleigh quotient under Hermitian constraints or symmetric constraints. These constraints depend on the structure of the rational matrix function and the parameter d and k in $G(z)$.

Due to the relation between the eigenvalues of rational matrix functions and the associated Rosenbrock system matrix, we also consider obtaining eigenvalue backward errors for the Rosenbrock system matrix. We provide two different approaches for two different measures to compute the eigenvalue backward error of the Rosenbrock system matrix, one by reducing it into an equivalent problem of optimizing sums of two generalized Rayleigh quotients and another via the structured μ -value of a constant matrix under block diagonal perturbations.

Eigenvalue backward errors of matrix pencils/polynomial are closely related to the distance to singularity, i.e., the distance to the nearest singular matrix pencil/polynomial. This is because if $A + zE$ is any regular matrix pencil, then the distance to the nearest singular matrix pencil is bounded from below by the infimum of the eigenvalue backward error of $A + zE$ over all the scalars $\lambda \in \mathbb{C}$. Given a regular matrix pencil $A + zE$, the problem of finding the distance to singularity has attracted much work in the literature and is still an open problem. If we consider additional structure on the coefficients matrices, then the problem of computing the structured distance to singularity (distance to the nearest singular pencil with the same structure) becomes more difficult and as far as we know, this problem with structure on the pencil has not been explored in the literature. We give a purely linear algebra-based approach to derive explicit computable formulas for the distance to the nearest structured pencil $\tilde{A} + z\tilde{E}$ having a common null vector. We then obtain a family of computable lower bounds for the unstructured and structured distances to singularity. Numerical experiments suggest that, in many cases, there is a significant difference between structured and unstructured distances. We also extend this approach to structured matrix polynomials with higher degrees.

सार

नॉनलीनर आइगेनवैल्यू प्रॉब्लम्स, विशेष रूप से पोलीनोमिअल और रैशनल आइगेनवैल्यू प्रॉब्लम्स, कई इंजीनियरिंग अनुप्रयोगों में हुई हैं और साहित्य में इनका व्यापक अध्ययन किया गया है। गणना किए गए आइगेनवैल्यू / आइगेनपेयर का स्ट्रक्चर्ड बैकवर्ड एरर विश्लेषण एल्गोरिदम की बैकवर्ड स्थिरता को जानने में महत्वपूर्ण भूमिका निभाता है जो संबंधित आइगेनवैल्यू प्रॉब्लम्स की स्ट्रक्चर को खोए बिना उनकी गणना करता है। स्ट्रक्चर्ड आइगेनपेयर बैकवर्ड एरर साहित्य में मैट्रिक्स पोलीनोमिअल और रैशनल मैट्रिक्स फंक्शन के लिए जानी जाती हैं। स्ट्रक्चर्ड आइगेनवैल्यू बैकवर्ड एरर कुछ स्ट्रक्चर के लिए मैट्रिक्स पोलीनोमिअल के लिए जानी जाती हैं, लेकिन रैशनल मैट्रिक्स फंक्शन की आइगेनवैल्यू बैकवर्ड एरर अभी भी अज्ञात हैं। कुछ स्ट्रक्चर्ड मैट्रिक्स पोलीनोमिअल और स्ट्रक्चर्ड मैट्रिक्स फंक्शन के लिए इन बैकवर्ड एरर का अध्ययन किया गया है।

हम पैरामीटर-निर्भर हर्मिटियन मैट्रिक्स के दूसरे सबसे बड़े आइगेनवैल्यू के संदर्भ में सिमेट्रिक मैट्रिक्स से जुड़े कुछ कंस्ट्रेंट्स के तहत रैले कौशैट को ऑप्टिमाइज़ करने के लिए स्पष्ट अभिव्यक्ति प्राप्त करते हैं, जिसने हमें स्क्यू-सिमेट्रिक, T-इवन, T-ओड, T-पैलिन्ड्रोमिक, और T-एंटीपैलिन्ड्रोमिक उच्च डिग्री मैट्रिक्स पोलीनोमिअल के लिए आइगेनवैल्यू बैकवर्ड एरर को प्राप्त करने की अनुमति दी है।

हम रैशनल मैट्रिक्स फंक्शन की स्ट्रक्चर्ड आइगेनवैल्यू बैकवर्ड एरर के लिए स्पष्ट गणना योग्य सूत्र प्राप्त करने के लिए [14,15] के विचारों का विस्तार करते हैं।

$$G(z) = \sum_{p=0}^d z^p A_p + \sum_{j=1}^k w_j(z) E_j$$

जहाँ गुणांक $(A_0, A_1, \dots, A_d, E_1, \dots, E_k) \in S \subseteq (\mathbb{C}^{n,n})^{d+k+1}$ मैट्रिसेस, और $w_j(z) = \frac{s_j(z)}{q_j(z)}$, जहाँ $s_j(z)$ और $q_j(z)$, के लिए $j = 1, 2, \dots, k$ कुछ सिमेट्रि गुण का अनुसरण करने वाले स्केलर पोलीनोमिअल हैं। इसके लिए हम आइगेनवैल्यू बैकवर्ड एरर समस्या को हर्मिटियन कंस्ट्रेंट्स या सिमेट्रिक कंस्ट्रेंट्स के तहत रेले कौशेंट को अधिकतम करने की एक समतुल्य समस्या में सुधारते हैं। ये कंस्ट्रेंट्स रैशनल मैट्रिक्स फ़ंक्शन की स्ट्रक्चर और $G(z)$ में पैरामीटर d और k पर निर्भर करती हैं।

रैशनल मैट्रिक्स फ़ंक्शनों के आइगेनवैल्यू और संबंधित रोसेनब्रॉक सिस्टम मैट्रिक्स के बीच संबंध के कारण, हम रोसेनब्रॉक सिस्टम मैट्रिक्स के लिए आइगेनवैल्यू बैकवर्ड एरर प्राप्त करने पर भी विचार करते हैं। हम रोसेनब्रॉक सिस्टम मैट्रिक्स की आइगेनवैल्यू बैकवर्ड एरर की गणना करने के लिए दो अलग-अलग उपायों के लिए दो अलग-अलग दृष्टिकोण प्रदान करते हैं, एक इसे दो सामान्यीकृत रेले कौशेंट के योग को अनुकूलित करने की एक समतुल्य समस्या में रिड्यूस करके और दूसरा ब्लॉक डायगोनल परटर्बेशन के तहत एक कांस्टेंट मैट्रिक्स के स्ट्रक्चर्ड μ -वैल्यू के माध्यम से।

मैट्रिक्स पेंसिल/पोलीनोमिअल की आइगेनवैल्यू बैकवर्ड एरर डिस्टेंस टू सिंगुलैरिटी से संबंधित हैं, अर्थात, निकटतम सिंगुलर मैट्रिक्स पेंसिल/पोलीनोमिअल की दूरी। ऐसा इसलिए है क्योंकि यदि $A + sE$ कोई रेगुलर मैट्रिक्स पेंसिल है, तो निकटतम सिंगुलर मैट्रिक्स पेंसिल की दूरी सभी स्केलर $\lambda \in \mathbb{C}$ पर $A + sE$ की आइगेनवैल्यू बैकवर्ड एरर के निम्नतम द्वारा नीचे से निर्धारित की जाती है। एक रेगुलर मैट्रिक्स पेंसिल $A + sE$ दिए जाने पर, डिस्टेंस टू सिंगुलैरिटी ज्ञात करने की समस्या ने साहित्य में बहुत काम किया है और यह अभी भी एक खुली समस्या है। यदि हम कोएफ़िशिएंट मैट्रिक्स पर अतिरिक्त स्ट्रक्चर पर विचार करते हैं, तो स्ट्रक्चर्ड डिस्टेंस टू सिंगुलैरिटी (समान स्ट्रक्चर के साथ निकटतम सिंगुलर पेंसिल की दूरी) की गणना करने की समस्या अधिक कठिन हो जाती है और जहाँ तक हम जानते हैं, मैट्रिक्स पेंसिल पर स्ट्रक्चर के साथ इस समस्या का साहित्य में पता नहीं लगाया गया है। हम एक कॉमन नल्ल स्पेस वाले निकटतम स्ट्रक्चर्ड पेंसिल $\tilde{A} + z\tilde{E}$ की दूरी के लिए स्पष्ट गणना योग्य सूत्र प्राप्त करने के लिए एक विशुद्ध रूप से रैखिक बीजगणित-आधारित दृष्टिकोण देते हैं। फिर हम सिंगुलैरिटी के लिए अनस्ट्रक्चर्ड और स्ट्रक्चर्ड दूरियों के लिए गणना योग्य निचली सीमाओं का एक परिवार प्राप्त करते हैं। संख्यात्मक प्रयोगों से पता चलता है कि, कई मामलों में, स्ट्रक्चर्ड और अनस्ट्रक्चर्ड दूरियों के बीच एक महत्वपूर्ण अंतर है। हम इस दृष्टिकोण को उच्च डिग्री वाले स्ट्रक्चर्ड मैट्रिक्स पोलीनोमिअल तक भी बढ़ाते हैं।

Contents

Certificate	i
Acknowledgements	iii
Abstract	v
List of Figures	xiii
List of Tables	xv
1 Introduction	1
1.1 Notation and terminology	4
1.2 Nonlinear eigenvalue problems	6
1.2.1 Polynomial eigenvalue problem	7
1.2.2 Rational eigenvalue problem	8
1.3 Backward error	12
1.3.1 Eigenpair backward error	12
1.3.2 Eigenvalue backward error	13
1.4 Rayleigh quotients	14
1.5 Organization of thesis	17
1.6 Preliminaries	19
1.6.1 Structured singular values	20

1.6.2	Mapping results	21
1.6.3	MATLAB programming tools	23
2	Optimizing Rayleigh quotient under symmetric constraints	27
2.1	Maximizing Rayleigh quotient with symmetric constraints	29
2.2	Applications: Computing eigenvalue backward errors for structured polynomials	37
2.2.1	T-palindromic and T-antipalindromic polynomials	38
2.2.2	T-even and T-odd polynomials	41
2.2.3	Skew-symmetric matrix polynomials	44
2.3	Numerical experiments	46
2.3.1	Comparison with the upper bound in [68]	46
2.3.2	Estimation to structured eigenvalue backward errors	47
3	Structured eigenvalue backward error of RMF	51
3.1	Preliminaries	52
3.2	Idea and framework	54
3.3	Symmetric and skew-symmetric structures	57
3.4	T-even/T-odd structures	60
3.5	Hermitian and related structures	64
3.6	Palindromic structure	68
3.7	Structured vs unstructured eigenvalue backward errors	77
4	Eigenvalue backward error of Rosenbrock system matrix	81
4.1	Backward error with full perturbation	82
4.2	Backward error with partial perturbation	85
4.2.1	Perturbing only one block of $S(z)$ at a time	86
4.2.2	Perturbing any two blocks of $S(z)$ at a time	90
4.2.3	Perturbing any three blocks of $S(z)$ at a time	97

5	Structured singular values and eigenvalue backward errors of system matrix	103
5.1	μ -values of rectangular matrices	105
5.2	Eigenvalue backward error of system matrix	111
5.2.1	Backward error with full perturbations	112
5.2.2	Backward error with partial perturbation	114
5.3	Numerical experiment	123
6	Structured distance to singularity	127
6.1	Preliminaries	128
6.2	Structured distances to a common null space	131
6.2.1	Unstructured pencils	131
6.2.2	Hermitian and related structures	134
6.2.3	Palindromic structures	136
6.2.4	Dissipative Hamiltonian structure	138
6.3	Structured distance to a common null space for higher degree polynomials	144
6.4	Structured distance to singularity	146
6.4.1	Unstructured distance to singularity	147
6.4.2	Hermitian and related structure	148
6.4.3	Palindromic structure	150
6.5	Unstructured vs. structured distances	151
	Conclusion	157
	Bibliography	161
	Publications	171
	Bio-Data	173

List of Figures

3.1	Eigenvalue perturbation curves for the Hermitian rational matrix function with respect to structured (left) and unstructured (right) perturbations. .	78
3.2	Eigenvalue perturbation curves for the *-palindromic RMF with respect to structured (left) and unstructured (right) perturbations.	80

List of Tables

1.1	Structure of the coefficient matrices of $G(z)$ in (1.1) and constraints on the rational functions $w_j(z)$	4
2.1	Estimation to $m_{hs_0s_1}$ from Theorem 2.1.4 and [68, Theorem 3.2.2]	47
2.2	Structured and unstructured eigenvalue backward errors for T-palindromic polynomial $P(z)$	48
2.3	Structured and unstructured eigenvalue backward errors for T-even polynomial $L(z)$	49
3.1	Structured and unstructured eigenvalue backward errors of a random Hermitian RMF.	79
6.1	Various structured distances for a regular DH pencil $sE + (J - R)$	143
6.2	Structured distance to a common null space for matrix polynomials	145
6.3	Structured eigenvalue backward errors for Hermitian related structures .	149
6.4	Various structured distances for Hermitian pencils	152
6.5	Various structured distances for *-palindromic pencils	153
6.6	Structured distances for DH pencils inspired by [59, Example 24]. The third and fourth columns are bounds for the distance $\delta_0^i(J, R, E)$	154
6.7	Structured distances for random DH pencils $zE - (J - R)$, where $J = 0$. The fourth and fifth columns are upper bounds for the distance $\delta_0^i(J, R, E)$. 154	

6.8	Structured distances for Mass-spring-damper model.	155
-----	--	-----