THE LIE STRUCTURE IN RINGS AND GROUP RINGS

by

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CERTIFICATE

This is to certify that the thesis entitled, 'THE LIE STRUCTURE IN RINGS AND GROUP RINGS' being submitted by Mr. Rajendra Kumar Sharma to the Indian Institute of Technology, Delhi, for the award of the degree of Doctor of Philosophy in Mathematics, is a record of bonafide research work carried out by him under my guidance and supervision for the last five years. To the best of my knowledge it has reached the standard, fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted, either in part or in full, to any other University or Institute for the award of any degree or diploma.

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ABSTRACT

This thesis is a study of certain aspects of the Lie structure in Rings and Group Rings. All rings considered are associative rings with identity $1 \neq 0$. Of course, Lie rings and Lie algebras will occur frequently throughout the thesis. In Chapter I we have studied the Lie structure in associative rings. Chapter II and Chapter IV deal with the associated Lie algebras of group algebras and the Lie structure in semi-prime and prime group rings respectively. In Chapter III we have studied the upper $\underline{\underline{X}}$-central chain of a Lie algebra where $\underline{\underline{X}}$ is any class of Lie algebras closed under taking subalgebras, quotients, and finite direct sums of its members.

Let $\mathcal{L}(R)$ denote the associated Lie ring of a ring R under the Lie multiplication $[x,y] = xy-yx$ for $x,y \in R$. Further suppose that the Lie ring $\mathcal{L}(R)$ is solvable of length n. It has been proved in Chapter I that if 3 is invertible in R, then the ideal J of R generated by all elements $[[[x_1,x_2],[x_3,x_4]],x_5]$ with $x_1,x_2,x_3,x_4,x_5 \in R$ is a nilpotent ideal of index atmost $(19\cdot 10^{n-3} - 1)\frac{2}{9}$ for all $n \geqslant 3$. Also if 2 and 3 are both invertible in R and $\mathcal{L}(R)$ is solvable, then the ideal of R generated by all elements $[x,y]$ with $x,y \in R$ has been proved to be a nil ideal of R. Also, if 2 and 3 are invertible in R and $m,n \geqslant 1$ are any two positive integers such that one of them m or n is odd then it is proved that $\gamma_m(\mathcal{L}(R))\cdot\gamma_n(\mathcal{L}(R)) \subseteq \gamma_{m+n-1}(\mathcal{L}(R))\cdot R$, where $\gamma_i(\mathcal{L}(R))$ denotes the ith term of the

lower central chain of $\mathcal{L}(R)$. As an application, we have proved that if 2 and 3 are invertible in R and R is a semi-prime ring with $\mathcal{L}(R)$ a solvable Lie ring, then R is commutative. Some applications to Lie solvable group rings are given in Chapter II. Also in the process we have done a detailed study of the lower central chain and the derived chain of an arbitrary Lie ideal of R. The results obtained are new and sometimes are improvements of certain known results. We have also given a large number of Lie identities as a tool for solving problems in this area. Group of units $\mathcal{U}(R)$ and Lie ideals which are also subrings of R, are studied. We have freely used the work of Jennings, Herstein and his collaborators, Gupta and Levin, and that of Zalesskii and Smirnov.

In Chapter II, we have studied the associated Lie algebra $\mathcal{L}(K[G])$ of the group algebra K[G] of the group G over the field K. We have given a complete characterization of those group algebras K[G] for which the associated Lie algebra $\mathcal{L}(K[G])$ is ideally finite as a Lie algebra. Also the question of when $\mathcal{L}(K[G])$ is ideally solvable is discussed. In the process we are able to give a new characterization of locally p-Abelian groups, p a prime. Using results of Chapter I, we have simplified some of the known results on Lie nilpotent and Lie solvable group rings. Groups of finite weight are also discussed. In this connection the work of Passi, Passman and Sehgal is particularly noteworthy. Here the standard material on group rings is taken from two books of Passman and also from Sehgal's book on the subject. Ideally finite Lie algebras are discussed in a monograph by Stewart. A book by Amayo and Stewart is also very helpful.

Let $\underline{\underline{X}}$ be a subalgebra, quotient, and finite direct sum closed class of Lie algebras. Further let L be any Lie algebra. Then we define $H(L:\underline{\underline{X}})$ to be all elements x belonging to L for which there exists an ideal I of L with $[x,I] = 0$ and L/I belongs to $\underline{\underline{X}}$. It has been proved that $H(L:\underline{\underline{X}})$ is an ideal of L satisfying many nice properties. Also $L/C_L(H(L:\underline{\underline{X}}))$ is proved to be residually-$\underline{\underline{X}}$. The upper $\underline{\underline{X}}$-central series $\{H_n(L:\underline{\underline{X}})\}$ has been defined and Lie algebras L for which $L = H_n(L:\underline{\underline{X}})$ have been studied.

Finally in the last Chapter IV, the Lie ideals and the Lie structure of semiprime and prime group algebras are studied. Some sporadic results are also given.

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