MULTIBODY MODELING AND HUNTING ANALYSIS OF RAIL VEHICLES

VISHNU SUKUMAR

DEPARTMENT OF MECHANICAL ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY DELHI

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by

VISHNU SUKUMAR

DEPARTMENT OF MECHANICAL ENGINEERING

submitted

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CERTIFICATE

This is to certify that the thesis titled **Multibody Modeling and Hunting Analysis of Rail Vehicles**, submitted by **Vishnu Sukumar**, to the Indian Institute of Technology, Delhi, for the award of the degree of **Doctor of Philosophy**, is a bona fide record of the research work done by him under our supervision. The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

Prof. S. P. Singh
Professor
Dept. of Mechanical Engineering
IIT-Delhi, 110016

Prof. S. K. Saha
Professor
Dept. of Mechanical Engineering
IIT-Delhi, 110016

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Place: New Delhi

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I dedicate my thesis to Almighty, who has blessed me with such wonderful experiences, surroundings, and people throughout my life.

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ABSTRACT

The unique profile of rail wheels imparts kinematic constraints on the motion of a wheelset. While such a profile allows a rail vehicle to negotiate a curving track, it also causes sustained vibrations called hunting. It is important to identify the critical hunting speed of a rail vehicle, which requires the integration of wheel-rail contact into the multibody formulation. In railway dynamics, there are two conventional approaches: (a) the elastic approach, in which wheelset has six degrees of freedom with respect to rails, and also wheels are could have penetration with rails. On the other hand, the constrained approach restricts penetration between wheels and rails.

All six degrees of freedom are uncoupled in most formulations that use an elastic approach. However, some motions could be coupled under certain conditions. For example, when wheels have a single-point contact with rails, the wheelset rotates about its roll center. Roll rotation also causes lateral translation. In this thesis, a novel approach is explored to represent the equivalent kinematic constraints of a wheelset. A linkage mechanism that is representative of a wheelset having point contact with rails, moving on a horizontal track is synthesized, by expanding on the concept of instantaneous roll center used in mechanics. Here, the identification of the wheelset's roll center is based on the shape of the wheel and rail profiles. Once the location of the roll center is known, it is shown that the dimensions of virtual links in the mechanism could be derived analytically. In the next step, the kinematic constraints of the linkage mechanism were derived using recursive relations of velocity level constraints using the Decoupled Natural Orthogonal Complement (DeNOC) matrices. In the beginning, multiple assumptions were made to obtain the system's governing equations of motion, notably, the use of knife-edge rails and conical wheels, constant roll center, etc. This led to the links in the mechanism having constant lengths. Moreover, the contact

parameters were also kept constant, which allowed us to obtain the governing equations of motion in a basic form that is available in textbooks and literature. The result of these simplifications is then demonstrated by analyzing the critical hunting speed of a suspended wheelset.

In the second portion of the thesis, several simplifications made to the linkage mechanism are removed, to simulate realistic scenarios. It was made possible using the concept of a wheel-rail contact triangle which enabled us to synthesize an instantaneous linkage mechanism for a wheelset. The kinematic parameters such as instantaneous roll center, and varying dimensions of virtual links for a wheelset undergoing hunting motion are illustrated. Ultimately, the dynamics simulation of the linkage mechanism is carried out by taking into account curved wheel and rail profiles. Hunting vibrations are transmitted to the bogies and carbody, causing them to shake excessively. This leads to undesirable ride quality. The latter part of this work involves the development of a serial chain multibody model of a railway carbody. Uneven wear between components could significantly reduce the critical hunting speed of a rail vehicle, and it was found that such variations were not previously taken into account in the hunting analysis. Thus, the multibody model of a railway carbody was subjected to an extensive parametric analysis by varying the properties of the individual components. Through this study, it was possible to systematically comprehend the critical hunting speed of the vehicle concerning the properties of individual components. Accordingly, design proposals are suggested to remedy adverse hunting vibrations.

The dissertation concludes with an investigation of the hunting behavior of a trainset with multiple carbody systems, using an equivalent coupler model that connects adjacent carbodies. By changing the attributes of the cars separately, the critical hunting speeds of two, three, and multi-car trainsets were examined. The hunting behavior of these systems was studied and the conditions under which the number of cars could influence the hunting behavior are listed.

Keywords: Railway dynamics, multibody systems, wheel-rail contact, hunting analysis.

सार

रेल पहियों की अनूठी प्रोफ़ाइल व्हीलसेट की गति (स्पीड) पर कीनेमेटीक्स कन्सट्रैन्ट का प्रदान करती है| जबकि ऐसी प्रोफ़ाइल एक रेल वाहन को कर्व ट्रैक पर चलने को संभव करता हैं, यह निरंतर कंपन (वाइब्रेशन) का कारण भी बनती है जिसे हंटिंग मोशन कहा जाता है| रेल वाहन की क्रिटिकल स्पीड की पहचान करना महत्वपूर्ण है, जिसके लिए मल्टीबॉडी फॉर्मूलेशन में व्हील-रेल संपर्क के एकीकरण की आवश्यकता होती है| रेलवे डायनामिक्स में, दो संभावनाएं (अप्रोच) हैं:| पहला है इलास्टिक अप्रोच, जिसमें व्हीलसेट में रेल के संबंध में 6 डिग्री ऑफ फ्रीडम (DOF) होती है| दूसरा है कन्स्ट्रैन्ड अप्रोच जो पहियों और रेलों के बीच प्रवेश को प्रतिबंधित करता है|

सभी DOF इलास्टिक अप्रोच का उपयोग करने वाले अधिकांश फॉर्मूलेशन में अयुग्मित होते हैं| हालाँकि, कुछ मोशन को कुछ शर्तों के तहत जोड़ा जा सकता है| उदाहरण के लिए, जब पहियों का रेल के साथ सिंगल-पॉइंट संपर्क होता है, तो व्हीलसेट अपने रोल सेंटर अक्ष के अनुदिश घूमता है| रोल रोटेशन भी लेटरल ट्रैन्स्लेशन का कारण बनता है| इस थीसिस में, व्हीलसेट के एक्विवैलेन्ट कीनेमेटीक्स कन्सट्रेन्ट्स का प्रतिनिधित्व करने के लिए एक उपन्यास दृष्टिकोण का पता लगाया गया है| एक लिंकेज मैकेनिज्म जो नाइफ-एज ट्रैक पर चलते हुए, रेल के साथ पॉइंट संपर्क वाले व्हीलसेट का प्रतिनिधि है, संश्लेषित किया गया है, यांत्रिकी में प्रयुक्त तात्कालिक रोल केंद्र की अवधारणा का विस्तार करके | यहां, व्हीलसेट के रोल सेंटर की पहचान व्हील और रेल प्रोफाइल के आकार पर आधारित है| एक बार रोल सेंटर का स्थान ज्ञात हो जाने पर, यह दिखाया जाता है कि मैकेनिज्म में वर्चुअल लिंक के आयाम विश्लेषणात्मक रूप से प्राप्त किए जा सकते हैं| अगले चरण में, लिंकेज मैकेनिज्म की वेलोसिटी कन्सट्रेन्ट्स को Decoupled Natural Orthogonal Complement (DeNOC) मैट्रिक्स का उपयोग करके वेग स्तर की बाधाओं के पुनरावर्ती संबंधों का उपयोग करके प्राप्त किया गया था। शुरुआत में, सिस्टम की इक्वैशन ऑफ मोशन को प्राप्त करने के लिए कई धारणाएँ बनाई गईं, विशेष रूप से, नाइफ-एज वाली रेल और कोनिकल पहियों का उपयोग, निरंतर रोल सेंटर, आदि। इससे मैकेनिज्म में कड़ियों (लिंक) की लंबाई स्थिर हो गई। इसके अलावा, संपर्क मापदंडों को भी सतत रखा गया, जिससे हमें इक्वैशन ऑफ मोशन को मूल रूप में प्राप्त करना संभव था जो पाठ्यपुस्तकों और साहित्य में उपलब्ध है। इन सरलीकरणों का परिणाम एक सस्पेन्डड व्हीलसेट की क्रिटिकल हंटिंग स्पीड का विश्लेषण करके प्रदर्शित किया जाता है|

थीसिस के दूसरे भाग में, यथार्थवादी परिदृश्यों का अनुकरण करने के लिए लिंकेज मैकेनिज्म में किए गए कई सरलीकरण हटा दिए गए हैं| इसे व्हील-रेल कांटेक्ट ट्रायंगल की अवधारणा का उपयोग करके संभव बनाया गया था, जिसने हमें व्हीलसेट के लिए इन्स्टन्टैनीअस लिंकेज मैकेनिज्म को संश्लेषित करने में सक्षम बनाया। इन्स्टन्टैनीअस रोल सेंटर और हंटिंग स्पीड से गुजरने वाले व्हीलसेट के लिए आभासी लिंक के विभिन्न आयामों जैसे गतिज मापदंडों को चित्रित किया गया है। अंततः, लिंकेज मैकेनिज्म की गतिशीलता का अनुकरण कर्व पहिया और रेल प्रोफाइल को ध्यान में रखकर किया जाता है।

हंटिंग के वाइब्रेशन बोगियों और कारबॉडी में संचारित होते हैं, जिससे वे अत्यधिक हिलते हैं | इससे सवारी की गुणवत्ता अवांछनीय हो जाती है| इस कार्य के उत्तरार्ध में रेलवे कारबॉडी के सीरियल चेन मल्टीबॉडी मॉडल का विकास शामिल है| घटकों के बीच असमान घिसाव एक रेल वाहन की क्रिटिकल हंटिंग स्पीड को काफी कम कर सकता है, और यह पाया गया कि हंटिंग एनालिसिस में इस तरह की विविधताओं को पहले ध्यान में नहीं रखा गया था। इस प्रकार, रेलवे कारबॉडी के मल्टीबॉडी मॉडल को अलग-अलग घटकों के गुणों को अलग-अलग करके व्यापक पैरामीट्रिक एनालिसिस के अधीन किया गया था | इस अध्ययन के माध्यम से, व्यक्तिगत घटकों के गुणों से संबंधित वाहन की क्रिटिकल हंटिंग स्पीड को व्यवस्थित रूप से समझना संभव था| तदनुसार, प्रतिकूल हंटिंग वाइब्रेशन को दूर करने के लिए डिज़ाइन प्रस्तावों का सुझाव दिया जाता है।

शोध प्रबंध एक एक्विवैलेन्ट कप्लर मॉडल का उपयोग करके, जो आसन्न कार्बोडी को जोड़ता है, एकाधिक कार्बोडी सिस्टम वाले ट्रेनसेट के हंटिंग व्यवहार की जांच के साथ समाप्त होता है। कारों की विशेषताओं को अलग-अलग बदलकर, दो, तीन और बहु-कार ट्रेनसेट की क्रिटिकल हंटिंग स्पीड की जांच की गई। इन प्रणालियों के हंटिंग व्यवहार का अध्ययन किया गया और उन परिस्थितियों को सूचीबद्ध किया गया जिनके तहत कारों की संख्या हंटिंग व्यवहार को प्रभावित कर सकती है।

कीवर्ड: रेलवे डाइनैमिक्स, मल्टीबॉडी सिस्टम, व्हील-रेल संपर्क, हंटिंग एनालिसिस

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ABBREVIATIONS

DOF	Degree(s) Of Freedom
DeNOC	Decoupled Natural Orthogonal Complement matrix
DAE	Differential Algebraic Equation
IITD	Indian Institute of Technology, Delhi
ODE	Ordinary Differential Equation
NOC	Natural Orthogonal Complement matrix
RMS	Root Mean Square
SBP	Smallest Bifurcation Point
2D	Two dimensional
3D	Three dimensional

NOTATION

Notations used in this thesis are mentioned here. Some rules, which are followed to classify the notations are listed below.

- Upper-case letters in bold, denote matrices.
- Lower-case letters in bold, are used for vectors.
- Points are mentioned using italic upper-case letters.
- Italic lower-case letters represent scalar quantities.

Accents

$(\tilde{\ })$	Skew symmetric form of a vector
$(\times)$	Cross product
$(\dot{\ })$	First time derivative (velocity)
$(\ddot{\ })$	Second time derivative (acceleration)
$(\hat{\ })$	Linearized matrices

Axes and coordinates

O	Origin of a frame
$\boldsymbol{x}$	Longitudinal axis
$\boldsymbol{y}$	Lateral axis
$\boldsymbol{z}$	Vertical axis
x	Longitudinal translation, m
y	Lateral translation, m
z	Vertical translation, m
ϕ	Roll angle, rad
θ	Pitch angle, rad
ψ	Yaw angle, rad

Greek symbols

Symbol	Description
α	Flange angle, rad
β_l	Contact angle of left wheel, rad
β_r	Contact angle of right wheel, rad
ϵ_x	Horizontal creepage
ϵ_y	Lateral creepage
γ	Wheelset conicity, rad
γ^w	Conicity of j^{th} wheelset, rad
$\boldsymbol{\lambda}$	Vector of Lagrange multipliers
λ	Conicity-track gauge ratio
ν	Coefficient of friction
ω_n	Hunting frequency, Hz
$\boldsymbol{\omega_i}$	Angular velocity vector of i^{th} link
ρ	Penetration depth, mm
φ_z	Spin creep about vertical axis
$\boldsymbol{\tau}$	External forces vector
$\boldsymbol{\tau^{wj}}$	Generalized force vector of j^{th} wheelset
$\boldsymbol{\tau^{bk}}$	Generalized force vector of the k^{th} bogie
$\boldsymbol{\tau^{cl}}$	Generalized force vector of the l^{th} carbody
$\boldsymbol{\tau^{cpl}}$	Generalized coupler force vector of the l^{th} carbody
$\dot{\boldsymbol{\Theta}}$	Vector of independent joint speeds

Latin letters

Symbol	Description
a	Half of track gauge, m
a_e	Semi-axis of contact ellipse along horizontal axis, mm
b_e	Semi-axis of contact ellipse along lateral axis, mm
$\boldsymbol{B_{i,i-1}}$	Twist propagation matrix
$\boldsymbol{\hat{C}^w}$	Linearized damping matrix of a wheelset
$\boldsymbol{\hat{C}^b}$	Linearized damping matrix of a wheelset
$\boldsymbol{\hat{C}^c}$	Linearized damping matrix of a carbody
$\boldsymbol{\hat{C}}$	Linearized damping matrix of a trainset
C_l	Contact point between left wheel, and left rail
C_r	Contact point between right wheel, and right rail
C_i	Center of mass of i^{th} link
$\dot{\boldsymbol{c}}_{\boldsymbol{i}}$	Linear velocity vector of i^{th} link
c_x^p	Longitudinal damping of primary suspension, Nm/sec
c_y^p	Lateral damping of primary suspension, Nm/sec
c_x^s	Longitudinal damping of secondary suspension, Nm/sec

c_y^s	Lateral damping of secondary suspension, Nm/sec
$\boldsymbol{D}$	Coefficient matrix
$\boldsymbol{d_i}$	Vector from origin to center of mass of i^{th} link
$\boldsymbol{d_i}$	Vector from center of mass of i^{th} link to the origin of $i+1^{st}$ link
$\boldsymbol{e_i}$	Joint axis vector of i^{th} joint
E_w	Young's modulus of wheel, N/m^2
E_r	Young's modulus of rail, N/m^2
$\boldsymbol{f_i}$	External force vector of i^{th} link
$\boldsymbol{F_c}$	Contact force vector
f_{cy}	Lateral contact force, N
f_{cz}	Vertical contact force, N
f_n	Normal contact force, N
f_t	Tangential contact force, N
f_{11}	Lateral creep force coefficient, N
f_{12}	Lateral/spin creep force coefficient, $N.m^2$
f_{22}	Spin creep force coefficient, N
f_{33}	Longitudinal creep force coefficient, N
g	Acceleration due to gravity, m/sec^2
G	Average shear modulus of wheel and rail, N/m^2
G_w	Shear modulus of wheel, N/m^2
G_r	Shear modulus of rail, N/m^2
$\boldsymbol{h}$	Vector of Convective Inertia
$\boldsymbol{I}$	Generalized Inertia Matrix of the multibody system
$\boldsymbol{I^{wj}}$	Generalized Inertia Matrix of j^{th} wheelset
$\boldsymbol{I^{bk}}$	Generalized Inertia Matrix of k^{th} bogie
$\boldsymbol{I^{c}}$	Generalized Inertia Matrix of carbody
$\boldsymbol{I_i}$	Inertia matrix of i^{th} link
I_x^w	Roll inertia of wheelset, $kg.m^2$
I_x^{wj}	Roll inertia of j^{th} wheelset, $kg.m^2$
I_x^{bk}	Roll inertia of k^{th} bogie, $kg.m^2$
I_z^w	Yaw inertia of wheelset, $kg.m^2$
I_z^{wj}	Yaw inertia of j^{th} wheelset, $kg.m^2$
I_z^{bk}	Yaw inertia of k^{th} bogie, $kg.m^2$
I_z^c	Yaw inertia of carbody, $kg.m^2$
i	Link number in the serial chain
j	Wheelset number
$\boldsymbol{J}$	Constraint Jacobian matrix
$\boldsymbol{\hat{K}^w}$	Linearized stiffness matrix of a wheelset
$\boldsymbol{\hat{K}^b}$	Linearized stiffness matrix of a wheelset
$\boldsymbol{\hat{K}^c}$	Linearized stiffness matrix of a carbody
$\boldsymbol{\hat{K}}$	Linearized stiffness matrix of a trainset
k_x^p	Longitudinal stiffness of primary suspension, N/m
k_y^p	Lateral stiffness of primary suspension, N/m
k_x^s	Longitudinal stiffness of secondary suspension, N/m
k_y^s	Lateral stiffness of secondary suspension, N/m
k	Bogie number
l_x^b	Length of bogie, m
l_y^b	Width of bogie, m

l_x^{bk}	Length of k^{th} bogie, m
l_y^{bk}	Width of k^{th} bogie, m
l_x^c	Length of carbody, m
l_y^c	Width of carbody, m
l_{1z}	Vertical coordinate (length) of first link, m
l_{2z}	Vertical coordinate (length) of second link, m
l_{3z}	Vertical coordinate (length) of third link, m
l_w	Wavelength of hunting oscillation, m
l	Carbody number
$\boldsymbol{M_i}$	Mass matrix of i^{th} link
$\boldsymbol{\hat{M}^w}$	Linearized mass matrix of a wheelset
$\boldsymbol{\hat{M}^b}$	Linearized mass matrix of a wheelset
$\boldsymbol{\hat{M}^c}$	Linearized mass matrix of a carbody
$\boldsymbol{\hat{M}}$	Linearized mass matrix of a trainset
m_w	Mass of wheelset, kg
m^{wj}	Mass of j^{th} wheelset, kg
m^{bk}	Mass of k^{th} bogie, kg
m^c	Mass of carbody, kg
m_e	Hertz contact coefficient
N^{MV}	Mean comfort index
n	Total number of links in a serial chain, also corresponds to # DOF
$\boldsymbol{n_i}$	External moment vector of i^{th} link
n_e	Hertz contact coefficient
$\boldsymbol{N}$	Natural Orthogonal Complement (NOC) Matrix
$\boldsymbol{N_l}$	Lower triangular Decoupled Natural Orthogonal Complement (DeNOC) Matrix
$\boldsymbol{N_d}$	Block diagonal Decoupled Natural Orthogonal Complement (DeNOC) Matrix
$\boldsymbol{n_l}$	Normal vector of left wheel-rail contact pair
$\boldsymbol{n_r}$	Normal vector of right wheel-rail contact pair
$\boldsymbol{p_l}$	Vector from wheelset C.G to left wheel-rail contact point
$\boldsymbol{p_r}$	Vector from wheelset C.G to right wheel-rail contact point
$\boldsymbol{p}$	State vector of a system
$\boldsymbol{p_i}$	Joint motion propagation vector of i^{th} joint
$\boldsymbol{Q}$	Rotation matrix
$\boldsymbol{Q_x}$	Rotation matrix about the longitudinal axis
$\boldsymbol{Q_y}$	Rotation matrix about the lateral axis
$\boldsymbol{Q_z}$	Rotation matrix about the vertical axis
$\boldsymbol{q}$	Vector of generalized coordinates
$\boldsymbol{q^{wj}}$	Generalized coordinates vector of j^{th} wheelset
$\boldsymbol{q^{bk}}$	Generalized coordinates vector of k^{th} bogie
$\boldsymbol{q^c}$	Generalized coordinates vector of carbody
r_l	Instantaneous rolling radius of left wheel, m
r_r	Instantaneous rolling radius of right wheel, m
r_0	Nominal rolling radius, m
r_0^w	Nominal rolling radius of j^{th} wheelset, m
$\boldsymbol{t_i}$	Twist vector of i^{th} joint
$\boldsymbol{t}$	Twist vector of the serial chain system
t	Simulation time, sec
t_k	Time period of hunting oscillation, sec

V	Forward velocity, m/sec
V_{cr}	Critical speed, m/sec
V_l	Velocity of left wheel at contact point, m/sec
V_{min}	Lower bound of forward velocity, m/sec
V_{max}	Upper bound of forward velocity, m/sec
V_{mid}	Average forward velocity, m/sec
V_r	Velocity of right wheel at contact point, m/sec
$\boldsymbol{W_i}$	Angular velocity matrix of i^{th} link
$\boldsymbol{w_i}$	Wrench vector of i^{th} link
$\boldsymbol{w^e}$	Vector of external forces
$\boldsymbol{w^c}$	Vector of constraint forces
$\bar{\mathbf{0}}$	Zero vector
$\mathbf{0}$	Zero matrix
$\mathbf{1}$	Identity matrix