JORDAN REGULAR UNITS IN RINGS AND GROUP RINGS

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by

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Dedicated to

My Family

Certificate

This is to certify that the thesis entitled **" Jordan Regular Units in Rings and Group Rings"** submitted by **Mrs. Parvesh Kumari** to the Indian Institute of Technology Delhi, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by her under my supervision. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results obtained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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Abstract

The aim of the thesis is to study the unit groups of rings and group rings and find presentations of general linear groups. In this thesis, we introduce the concept of Jordan regular elements and Jordan regular units. An element a of a ring R is called a Jordan regular element if there exist an idempotent e and a unit u in R such that $a = eu + ue$. A Jordan regular element which is also a unit is called a Jordan regular unit. In this thesis, we study Jordan regular elements in rings, fields and group rings. Set of Jordan regular elements in any ring, field and group ring is always non-empty, because 0 is there. We have shown that if 2 is a unit in R, where R is a commutative ring with unity, then every unit in $M(n, R)$ is a Jordan regular unit, but if 2 is not a unit in R, then this need not be true. Set of non Jordan regular units in $M(2, \mathbb{Z}_n)$ is a subgroup of $\mathcal{U}(M(2, \mathbb{Z}_n))$, for even n. If n is a power of 2, then we establish that the order of the set of Jordan regular units in $M(2, \mathbb{Z}_n)$ is half of the order of $\mathcal{U}(M(2, \mathbb{Z}_n))$. In $GL(2, F)$, where F is a finite field of characteristic 2, every non scalar matrix, having same diagonal elements is a Jordan regular element. We also prove that the group algebra KG of a group G over a field K of characteristic 2 does not have Jordan regular units. Further, we have proved that $M(3, \mathbb{Z}_{2n})$ does not contain Jordan regular units. If F is a finite field of characteristic 2, then also $M(3, F)$ does not contain any Jordan regular units.

We have determined Jordan regular generators of $GL(2, F_{2^n})$ and obtained presentations of $GL(2, F_{2^n})$, for $n = 2,\ 3,\ 4,\ 5$. Presentations of linear groups $GL(2, \mathbb{Z}_{16})$, $GL(2, \mathbb{Z}_{18})$, $GL(2, \mathbb{Z}_{20})$, $GL(2, \mathbb{Z}_{24})$, $GL(2, \mathbb{Z}_{28})$, $GL(2, \mathbb{Z}_{30})$, $GL(2, \mathbb{Z}_{32})$, $GL(2, \mathbb{Z}_{36})$, $GL(2, \mathbb{Z}_{38})$, $GL(2, \mathbb{Z}_{40})$ and $GL(2, \mathbb{Z}_{42})$ are also given having Jordan regular units as generators.

Further, we have given a complete characterization of $\mathcal{U}(\mathbb{Z}_2 D_{10})$.

सारॉंश

इस शोध प्रबंध का उद्देश्य रिंग्स और ग्रुप रिंग्स के यूनिट ग्रुप का अध्ययन और जनरल लीनियर ग्रुप्स की प्रेजेंटेशन निकालना है| इस शोध प्रबंध में हमने जॉर्डन रेगुलर एलिमेंट्स और जॉर्डन रेगुलर यूनिट्स के कांसेप्ट का परिचय कराया है| रिंग R का एक एलिमेंट a, जॉर्डन रेगुलर एलिमेंट कहलाता है, यदि R में एक आइडेम्पोटेंट e और एक यूनिट u उपस्थित हों, इस प्रकार कि $a = eu + ue$ हो जाए। एक जॉर्डन रेगुलर एलिमेंट यदि एक यूनिट भी है, तो यह जॉर्डन रेगुलर यूनिट कहलाता है| इस शोध प्रबंध में हमने जॉर्डन रेगुलर एलिमेंट्स का रिंग्स, फील्ड्स और ग्रुप रिंग्स में अध्ययन किया है| जॉर्डन रेगुलर एलिमेंट्स का सेट किसी भी रिंग, फील्ड और ग्रुप रिंग में हमेशा नॉन-एम्प्टी है, क्यूंकि वहां शून्य उपस्थित है। हमने दर्शाया है कि यदि 2, रिंग R में यूनिट है, जहाँ R यूनिटी रखने वाला एक कम्यूटेटिव रिंग है, तब $M(n, R)$ की प्रत्येक यूनिट जॉर्डन रेगुलर यूनिट है, परन्तु यदि 2, रिंग R में यूनिट नहीं है, तब यह सत्य हो जरूरी नहीं है| प्रत्येक n सम के लिए $M(2,\mathbb{Z}_n)$ में नॉन जॉर्डन रेगुलर यूनिट्स , $\mathcal{U}(M(2 , \mathbb{Z}_n))$ का एक सबग्रुप है| यदि संख्या n, 2 की एकघात है, तब हमने स्थापित किया है कि $M(2 , \mathbb{Z}_n)$ में जॉर्डन रेगुलर यूनिट्स के सेट का आर्डर $\mathcal{U}$ $(M(2, \mathbb{Z}_n))$ के आर्डर का आधा है| ग्रुप GL (2, $\mathbb{F}$) में, प्रत्येक नॉन स्केलर आव्यूह जिसमें डायगोनल के सभी एलिमेंट समान हैं, एक जॉर्डन रेगुलर एलिमेंट है, जहाँ $\mathbb{F}$ कैरेक्टरिस्टिक 2 का एक फाइनाइट फील्ड हैं| हमने यह भी दिखाया है कि एक फील्ड जिसकी कैरेक्टरिस्टिक 2 है, पर ग्रुप अलजेब्रा KG में कोई जॉर्डन रेगुलर यूनिट्स नहीं है| तदुपरांत, हमने दर्शाया है कि $M(3, \mathbb{Z}_{2n})$ में कोई जॉर्डन रेगुलर यूनिट्स नहीं है| यदि $\mathbb{F}$ कैरेक्टरिस्टिक 2 का एक फाइनाइट फील्ड है, तब भी $M(3, \mathbb{F})$ में कोई जॉर्डन रेगुलर यूनिट नहीं है।

हमने GL (2, $\mathbb{F}_{2^n}$) के जॉर्डन रेगुलर जनरेटर्स ज्ञात किये हैं, और n =2, 3, 4, 5 के लिए GL(2, $\mathbb{F}_{2^n}$) की प्रेजेंटेशंस दी हैं| लीनियर ग्रुप्स GL (2, $\mathbb{F}_{16}$), GL (2, $\mathbb{F}_{18}$), GL (2, $\mathbb{F}_{20}$), GL (2, $\mathbb{F}_{24}$), GL (2, $\mathbb{F}_{28}$), GL (2, $\mathbb{F}_{30}$), GL (2, $\mathbb{F}_{32}$), GL (2, $\mathbb{F}_{36}$), GL (2, $\mathbb{F}_{38}$), GL (2, $\mathbb{F}_{40}$), और GL (2, $\mathbb{F}_{42}$) की प्रेजेंटेशंस, जहाँ जनरेटर्स जॉर्डन रेगुलर यूनिट्स हैं, ज्ञात की है|

तदउपरांत, हमने U ($\mathbb{Z}_2 D_{10}$) का पूरा कैरक्टराइज़ेशन ज्ञात किया है|

Contents

List of Symbols

In the following notations, X is a set, G is a group, R is a ring and F is a field.

Symbol	**Meaning**
$\mathbb{N}$	the set of natural numbers
$\mathbb{Q}$	the field of rational numbers
$\mathbb{R}$	the field of real numbers
$\mathbb{Z}$	the ring of integers
$\mathbb{Z}_n$	the ring of integers modulo n
$\forall$	for all
$\in$	belongs to
$\notin$	does not belong to
$\subseteq$	subset or equal
$\subsetneq$	proper subset
$\cup$	union
$\cap$	intersection
$X \setminus E$	the complement of E in X
$\emptyset$	empty set

$\square$	end of a proof
$\lvert X\rvert$	cardinality of the set X
$\cong$	is isomorphic to
ϕ	Euler's totient function
(x, y)	$x^{-1}y^{-1}xy$
$\mathcal{Z}(G)$	center of G
$H \leq G$	H is a subgroup of G
$H \trianglelefteq G$	H is a normal subgroup of G
$[G : H]$	index of H in G
G/H	quotient of G by H
$\langle Y\rangle, \langle x\rangle$	subgroup of G generated by $Y \subseteq G$ and by the element $x \in G$
$o(x)$	order of $x \in G$
C_n	the cyclic group of order n
D_{2n}	the dihedral group of order $2n$
$\mathcal{U}(R)$	the unit group of R
$charR$	characteristic of R
$\det(A)$	determinant of matrix A
$M(n, R)$	the ring of $n \times n$ matrices over R
$GL(n, R)$	general linear group of degree n over R
$SL(n, R)$	special linear group of degree n over R
F^*	$F \setminus \{0\}$
RG	group ring of the group G over the ring R
$\mathcal{U}(RG)$	unit group of the group ring RG