

GRAPH COLORING AND ITS VARIATIONS: STRUCTURAL AND ALGORITHMIC STUDY

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by

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Dedicated to
My Family

Certificate

This is to certify that the thesis entitled “**Graph coloring and its variations: Structural and algorithmic study**” submitted by “**Ms. Priyamvada**” to the **Indian Institute of Technology Delhi**, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bona fide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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Abstract

With its origin embedded in attempts to solve the famous Four Color Problem, graph coloring has flourished into one of the most important and well studied areas of current research. Graph coloring, in general, is an assignment of colors, integers from $\{1, 2, \dots, k\}$, to the vertex set or edge set of a graph. Various variations of graph coloring have been defined and studied in literature, owing to their practical significance. In this thesis, we study the structural and algorithmic aspects of four variations of graph coloring, namely, (i) neighbor-sum-2-distinguishing edge-weighting, (ii) injective coloring, (iii) exact square coloring, and (iv) injective edge coloring.

A *neighbor-sum-distinguishing W -edge-weighting* of a graph $G = (V, E)$ is an assignment ω of weights from a set of integers W to the set of edges E of G such that for every pair of adjacent vertices, the incident sums induced by the edge-weighting are different, where the incident sum of a vertex v induced by the edge-weighting ω is $\sigma(v) = \sum_{u \in N(v)} \omega(uv)$, where $N(v)$ is the set of neighbors of v in G . Whereas, a *neighbor-sum-2-distinguishing W -edge-weighting* of G is a neighbor-sum-distinguishing W -edge-weighting such that the incident sums of every pair of adjacent vertices differ by at least 2. A recently proposed conjecture about neighbor-sum-2-distinguishing $\{1, 3, 5\}$ -edge-weighting states that any graph with no component isomorphic to K_2 admits a neighbor-sum-2-distinguishing

$\{1, 3, 5\}$ -edge-weighting. In this thesis, we prove that deciding whether there exists a neighbor-sum-2-distinguishing $\{1, 3\}$ -edge-weighting is NP-complete for bipartite graphs. We present an algorithm that computes a neighbor-sum-2-distinguishing $\{1, 3\}$ -edge-weighting of the central graph of any graph in polynomial time and thus prove the above conjecture for the central graph of any graph. We study another aspect of neighbor-sum-2-distinguishing edge-weighting of a graph G , namely, the minimum number of incident sums used by a neighbor-sum-2-distinguishing $\{1, 3\}$ -edge-weighting, which is denoted by $\gamma_{\Sigma>1,\{1,3\}}(G)$. We prove that the problems of deciding whether there exists a neighbor-sum-2-distinguishing $\{1, 3\}$ -edge-weighting of G from a given set of sums, and deciding whether $\gamma_{\Sigma>1,\{1,3\}}(G) \leq k$ are NP-complete for bipartite graphs. However, we provide both upper and lower bounds on $\gamma_{\Sigma>1,\{1,3\}}(C(G))$ for central graphs of bipartite graphs and split graphs. We also propose an algorithm that computes the optimal neighbor-sum-2-distinguishing $\{1, 3\}$ -edge-weighting of the central graphs of cycles and paths in polynomial time.

A vertex coloring of a graph $G = (V, E)$ that uses k colors is called an *injective k -coloring* of G if no two vertices having a common neighbor have the same color. The minimum k for which G has an *injective k -coloring* is called the *injective chromatic number* of G . Given a graph G and a positive integer k , the DECIDE INJECTIVE COLORING PROBLEM is to decide whether G admits an injective k -coloring. It is known that DECIDE INJECTIVE COLORING PROBLEM is NP-complete for bipartite graphs. In this thesis, we strengthen this result by proving that this problem remains NP-complete for perfect elimination bipartite graphs, star-convex bipartite graphs and comb-convex bipartite graphs, which are proper subclasses of bipartite graphs. Moreover, we show that for every $\epsilon > 0$, it is not possible to efficiently approximate the injective chromatic number of a perfect elimination bipartite graph within a factor of $n^{\frac{1}{3}-\epsilon}$ unless ZPP = NP. However, we propose a linear time algorithm for biconvex bipartite graphs and $O(nm)$ time algorithm for convex bipartite graphs for finding the optimal injective coloring. We prove that the injective chromatic number

of a chordal bipartite graph can be determined in polynomial time. It is known that DECIDE INJECTIVE COLORING PROBLEM is NP-complete for chordal graphs. We give linear time algorithms for computing the injective chromatic number of proper interval graphs and threshold graphs, which are proper subclasses of chordal graphs. DECIDE INJECTIVE COLORING PROBLEM is also known to be NP-complete for split graphs. We show that DECIDE INJECTIVE COLORING PROBLEM remains NP-complete for $K_{1,t}$ -free split graphs for $t \geq 4$ and polynomially solvable for $t \leq 3$.

A vertex coloring of a graph $G = (V, E)$ is called an *exact square coloring* of G if any two vertices at distance exactly 2 receive different colors. The minimum number of colors required by an exact square coloring is called the *exact square chromatic number* of G and is denoted by $\chi^{[\#2]}(G)$. Given a graph G and a positive integer k , the EXACT SQUARE COLORING PROBLEM is to decide whether G admits an exact square coloring using k colors. It is known that EXACT SQUARE COLORING PROBLEM is NP-complete for chordal graphs. In this thesis, we strengthen this result by proving that this problem remains NP-complete for undirected path graphs, which is a proper subclass of chordal graphs. However, we give linear time algorithms for computing the exact square chromatic number of proper interval graphs and threshold graphs. Moreover, for a proper interval graph G , we show that $\chi^{[\#2]}(G) \leq 3$. We also propose a polynomial time algorithm to produce an exact square coloring of a block graph G using at most $\chi^{[\#2]}(G) + 1$ colors. Next, we study a lower bound of $\chi^{[\#2]}(G)$. A subset S of vertices of a graph $G = (V, E)$ is called an *exact square clique* of G if the distance between any two vertices in S is exactly 2. The cardinality of the maximum exact square clique of G is called the exact square clique number of G and is denoted by $\omega^{[\#2]}(G)$. Clearly, $\omega^{[\#2]}(G) \leq \chi^{[\#2]}(G)$. Given a graph G and a positive integer k , the problem of deciding whether $\omega^{[\#2]}(G)$ is at least k , is known to be NP-complete for bipartite graphs and chordal graphs. In this thesis, we strengthen these results by proving that this problem remains NP-complete for undirected path

graphs, perfect elimination bipartite graphs, star-convex bipartite graphs and comb-convex bipartite graphs. We also compute the exact value of $\omega^{[\#2]}(G)$ for proper interval graphs, threshold graphs, block graphs and convex bipartite graphs.

An *injective k -edge-coloring* of a graph $G = (V, E)$ is an assignment $\omega : E \rightarrow \{1, 2, \dots, k\}$ of colors to the edges of G such that any two edges e and f receive distinct colors if there exists an edge $g = xy$ different from e and f such that e is incident on x and f is incident on y . The minimum value of k for which G admits an injective k -edge-coloring is called the *injective chromatic index* of G and is denoted by $\chi'_i(G)$. Clearly, an injective edge coloring is the natural edge-version of the notion of an *injective coloring*. Given a graph G and a positive integer k , the INJECTIVE EDGE COLORING PROBLEM is to decide whether G admits an injective k -edge-coloring. It is known that INJECTIVE EDGE COLORING PROBLEM is NP-complete for general graphs. In this thesis, we strengthen this result by proving that INJECTIVE EDGE COLORING PROBLEM is NP-complete for bipartite graphs by proving that this problem remains NP-complete for perfect elimination bipartite graphs and star-convex bipartite graphs. However, we propose linear time algorithms for computing the injective chromatic index of trees and chain graphs, which is a proper subclass of both perfect elimination bipartite graphs and star-convex bipartite graphs. We also propose linear time algorithm for computing the injective chromatic index of threshold graphs. A graph G is known as a *perfect EIC-graph* if $\chi'_i(G)$ is equal to the number of edges in a maximum clique of G . We prove that threshold graphs are perfect EIC-graphs.

Various operations defined over graphs form an important method to both construct new graphs and structurally characterize particular graph classes. In this thesis, we have also studied these variations of graph coloring under some standard graph operations and products, including Cartesian product, strong product, lexicographic product, corona product, edge corona product, join, subdivision and Mycielskian of a graph.

सार

प्रसिद्ध चार रंग समस्या को हल करने के प्रयासों में इसकी उत्पत्ति के साथ, ग्राफ कलरिंग वर्तमान शोध के सबसे महत्वपूर्ण और अच्छी तरह से अध्ययन किए गए क्षेत्रों में से एक में विकसित हुआ है। ग्राफ कलरिंग, सामान्य रूप से, रंगों का एक असाइनमेंट है, $\{1, 2, \dots, k\}$ से पूर्णांक, ग्राफ के वर्टेक्स सेट या एज सेट तक। उनके व्यावहारिक महत्व के कारण, ग्राफ कलरिंग के विभिन्न रूपों को साहित्य में परिभाषित और अध्ययन किया गया है। इस थीसिस में, हम ग्राफ कलरिंग के चार रूपों के संरचनात्मक और एल्गोरिथम पहलुओं का अध्ययन करते हैं, अर्थात्, (i) पड़ोसी-योग-2-विभेदक किनारे-भार, (ii) इंजेक्शन कलरिंग, (iii) सटीक वर्ग रंग, और (iv) इंजेक्शन एज कलरिंग।

एक ग्राफ $G = (V, E)$ का एक पड़ोसी-योग-विभेदित W -किनारे-भार, पूर्णांक W के एक सेट से G के किनारों E के सेट तक भार का एक असाइनमेंट है, जैसे कि आसन्न कोने की प्रत्येक जोड़ी के लिए, किनारे-भार द्वारा प्रेरित घटना योग भिन्न होते हैं, जहां किनारे-भार ω द्वारा प्रेरित एक शीर्ष v का घटना योग $\sigma(v) = \sum_{u \in N(v)} \omega(uv)$ होता है, जहां $N(v)$ G में v के पड़ोसियों का समुच्चय है। जबकि, G का एक पड़ोसी-योग-2-विभेदित W -किनारे-भार एक पड़ोसी-योग-विभेदकारी W -किनारे-भार है, जिससे घटना के प्रत्येक जोड़े का योग होता है आसन्न कोने कम से कम 2 से भिन्न होते हैं। पड़ोसी-योग-2-भेद $\{1, 3, 5\}$ -एज-वेटिंग के बारे में हाल ही में प्रस्तावित अनुमान में कहा गया है कि K_2 के लिए कोई घटक आइसोमॉर्फिक वाला कोई भी ग्राफ एक पड़ोसी-योग-2-विभेद $\{1, 3, 5\}$ -किनारे-भार को स्वीकार करता है। इस थीसिस में, हम यह साबित करते हैं कि यह तय करना कि क्या कोई पड़ोसी-योग-2-विभेद $\{1, 3\}$ -एज-वेटिंग मौजूद है, द्विदलीय ग्राफ के लिए एनपी-पूर्ण है। हम एक एल्गोरिथम प्रस्तुत करते हैं जो बहुपद समय में किसी भी ग्राफ के केंद्रीय ग्राफ के पड़ोसी-योग-2-भेद $\{1, 3\}$ -किनारे-भार की गणना करता है और इस प्रकार किसी भी ग्राफ के केंद्रीय ग्राफ के लिए उपरोक्त अनुमान को साबित करता है। हम एक ग्राफ G के पड़ोसी-योग-2-विभेदक किनारे-भार के एक अन्य पहलू का अध्ययन करते हैं, अर्थात्, पड़ोसी-योग-2-भेद $\{1, 3\}$ -किनारे-भार द्वारा उपयोग किए जाने वाले घटना योगों की न्यूनतम संख्या, जो है $\gamma_{\Sigma > 1, \{1, 3\}}(G)$ द्वारा निरूपित। हम साबित करते हैं कि यह तय करने की समस्याएं मौजूद हैं कि क्या एक पड़ोसी-योग-2-भेद $\{1, 3\}$ -किनारे-भार के एक दिए गए सेट से है, और यह तय करना कि क्या $\gamma_{\Sigma > 1, \{1, 3\}}(G) \leq k$ द्विदलीय रेखांकन के लिए NP-पूर्ण हैं। हालाँकि, हम द्विदलीय ग्राफ और विभाजित ग्राफ के केंद्रीय ग्राफ के लिए $\gamma_{\Sigma > 1, \{1, 3\}}(C(G))$ पर ऊपरी और निचली दोनों सीमाएँ प्रदान करते हैं। हम एक एल्गोरिथम भी प्रस्तावित करते हैं जो बहुपद समय में चक्रों और पथों के केंद्रीय रेखांकन के इष्टतम पड़ोसी-योग-2-विभेद $\{1, 3\}$ -किनारे-भार की गणना करता है।

ग्राफ $G = (V, E)$ का एक शीर्ष रंग जो k रंगों का उपयोग करता है, G का एक इंजेक्टिव k -रंग कहलाता है यदि एक समान पड़ोसी वाले दो शीर्षों का रंग समान नहीं है। न्यूनतम k जिसके लिए G में एक इंजेक्टिव k -रंग है, G की इंजेक्टिव क्रोमैटिक

संख्या कहलाती है। एक ग्राफ G और एक धनात्मक पूर्णांक k को देखते हुए, डिसाइड इंजेक्टिव कलरिंग समस्या यह तय करना है कि G एक इंजेक्टिव k -रंग को स्वीकार करता है या नहीं। यह ज्ञात है कि डिसाइड इंजेक्टिव कलरिंग प्रॉब्लम द्विदलीय ग्राफ के लिए एनपी-पूर्ण है। इस थीसिस में, हम यह साबित करके इस परिणाम को मजबूत करते हैं कि यह समस्या पूर्ण उन्मूलन द्विदलीय ग्राफ, स्टार-उत्तल द्विदलीय ग्राफ और कंधी-उत्तल द्विदलीय ग्राफ के लिए एनपी-पूर्ण बनी हुई है, जो द्विदलीय ग्राफ के उचित उपवर्ग हैं। इसके अलावा, हम दिखाते हैं कि प्रत्येक $\epsilon > 0$ के लिए, $n^{\frac{1}{3}-\epsilon}$ के एक कारक के भीतर एक पूर्ण उन्मूलन द्विदलीय ग्राफ के इंजेक्टिव रंगीन संख्या को कुशलतापूर्वक अनुमानित करना संभव नहीं है, जब तक कि $ZPP = NP$ न हो। हालांकि, हम इष्टतम इंजेक्टिव रंग खोजने के लिए उभयलिंगी द्विदलीय ग्राफ और उत्तल द्विदलीय ग्राफ के लिए $O(nm)$ समय एल्गोरिथम के लिए एक रैखिक समय एल्गोरिथम का प्रस्ताव करते हैं। हम सिद्ध करते हैं कि एक कॉर्डल द्विदलीय ग्राफ की अंतःक्षेपी वर्णिक संख्या बहुपद समय में निर्धारित की जा सकती है। यह ज्ञात है कि कॉर्डल ग्राफ के लिए डिसाइड इंजेक्टिव कलरिंग प्रॉब्लम एनपी-कम्प्लीट है। हम उचित अंतराल ग्राफ और थ्रेशोल्ड ग्राफ के इंजेक्टिव क्रोमैटिक नंबर की गणना के लिए रैखिक समय एल्गोरिदम देते हैं, जो कॉर्डल ग्राफ के उचित उपवर्ग हैं। डिसाइड इंजेक्टिव कलरिंग प्रॉब्लम को स्प्लिट ग्राफ के लिए एनपी-कम्प्लीट के रूप में भी जाना जाता है। हम दिखाते हैं कि डिसाइड इंजेक्टिव कलरिंग प्रॉब्लम $K_{1,t}$ फ्री स्प्लिट ग्राफ, $t \geq 4$ के लिए NP-पूर्ण बनी हुई है, और $t \leq 3$ के लिए बहुपद रूप से सॉल्व करने योग्य है।

ग्राफ $G = (V, E)$ के शीर्ष रंग को G का सटीक वर्गाकार रंग कहा जाता है यदि ठीक 2 की दूरी पर स्थित किन्हीं दो शीर्षों को अलग-अलग रंग प्राप्त होते हैं। एक सटीक वर्ग रंग के लिए आवश्यक रंगों की न्यूनतम संख्या को G का सटीक वर्ग वर्णिक संख्या कहा जाता है और इसे $\chi^{[2]}(G)$ द्वारा दर्शाया जाता है। एक ग्राफ G और एक धनात्मक पूर्णांक k को देखते हुए, सटीक वर्ग रंग समस्या यह तय करना है कि क्या G k रंगों का उपयोग करके एक सटीक वर्ग रंग स्वीकार करता है। यह ज्ञात है कि कॉर्डल ग्राफ के लिए सटीक वर्ग रंग समस्या एनपी-पूर्ण है। इस थीसिस में, हम यह साबित करके इस परिणाम को मजबूत करते हैं कि यह समस्या अप्रत्यक्ष पथ रेखांकन के लिए एनपी-पूर्ण बनी हुई है, जो कि कॉर्डल ग्राफ का एक उचित उपवर्ग है। हालांकि, हम उचित अंतराल ग्राफ और थ्रेशोल्ड ग्राफ की सटीक वर्ग रंगीन संख्या की गणना के लिए रैखिक समय एल्गोरिदम देते हैं। इसके अलावा, एक उचित अंतराल ग्राफ G के लिए, हम दिखाते हैं कि $\chi^{[2]}(G) \leq 3$ । हम एक बहुपद समय एल्गोरिथम का भी प्रस्ताव करते हैं जो ब्लॉक ग्राफ G का एक सटीक वर्ग रंग बनाने के लिए अधिकतम $\chi^{[2]}(G) + 1$ रंग का उपयोग करता है। इसके बाद, हम $\chi^{[2]}(G)$ की निचली सीमा का अध्ययन करते हैं। ग्राफ $G = (V, E)$ के शीर्षों का एक उपसमुच्चय S , G का एक सटीक वर्ग समूह कहलाता है, यदि S में किन्हीं दो शीर्षों के बीच की दूरी ठीक 2 है। G के अधिकतम सटीक वर्ग समूह की कार्डिनैलिटी को सटीक कहा जाता है G का वर्ग समूह संख्या और $\omega^{[2]}(G)$ द्वारा निरूपित किया जाता है। स्पष्ट रूप से, $\omega^{[2]}(G) \leq \chi^{[2]}(G)$ । एक ग्राफ G और एक धनात्मक पूर्णांक k को देखते हुए, यह तय करने की समस्या कि क्या $\omega^{[2]}(G)$ कम से कम k है, को द्विदलीय ग्राफ और कॉर्डल ग्राफ के लिए NP-पूर्ण माना जाता है। इस थीसिस में, हम इन परिणामों को यह साबित करके मजबूत करते हैं कि यह समस्या अप्रत्यक्ष पथ के लिए एनपी-पूर्ण बनी हुई है रेखांकन, पूर्ण उन्मूलन द्विदलीय रेखांकन, तारा-उत्तल द्विदलीय रेखांकन और संयुक्त उत्तल द्विदलीय रेखांकन। हम उचित अंतराल ग्राफ, थ्रेशोल्ड ग्राफ, ब्लॉक ग्राफ और उत्तल द्विदलीय ग्राफ के लिए $\omega^{[2]}(G)$ के सटीक मान की गणना भी करते हैं।

ग्राफ $G = (V, E)$ का एक इंजेक्शन k -किनारे-रंग एक असाइनमेंट है $\omega : E \rightarrow \{1, 2, \dots, k\}$ रंगों का G के किनारों पर इस तरह कि कोई भी दो किनारे e और f अलग-अलग रंग प्राप्त करें यदि कोई किनारा मौजूद है $g = xy$, e और f से भिन्न है जैसे कि e, x पर घटना है और f, y पर आपतित है। k का न्यूनतम मान जिसके लिए G एक इंजेक्शन k -किनारे-रंग को स्वीकार करता है, G का इंजेक्टिव क्रोमैटिक इंडेक्स कहलाता है और इसे $\chi'_i(G)$ द्वारा दर्शाया जाता है। स्पष्ट रूप से, एक इंजेक्शन किनारे का रंग एक इंजेक्शन रंग की धारणा का प्राकृतिक किनारा-संस्करण है। एक ग्राफ G और एक धनात्मक पूर्णांक k को देखते हुए, इंजेक्टिव एज कलरिंग समस्या यह तय करना है कि G एक इंजेक्टिव k -एज-कलरिंग को स्वीकार करता है या नहीं। यह ज्ञात है कि सामान्य रेखांकन के लिए इंजेक्टिव एज कलरिंग समस्या एनपी-पूर्ण है। इस थीसिस में, हम यह साबित करके इस परिणाम को मजबूत करते हैं कि द्विदलीय ग्राफ के लिए इंजेक्शन एज कलरिंग समस्या एनपी-पूर्ण है, यह साबित करके कि यह समस्या पूर्ण उन्मूलन द्विदलीय ग्राफ और स्टार-उत्तल द्विदलीय ग्राफ के लिए एनपी-पूर्ण बनी हुई है। हालांकि, हम पेड़ों और चेन ग्राफ के इंजेक्शन क्रोमैटिक इंडेक्स की गणना के लिए रैखिक समय एल्गोरिदम का प्रस्ताव करते हैं, जो कि पूर्ण उन्मूलन द्विदलीय ग्राफ और स्टार-उत्तल द्विदलीय ग्राफ दोनों का एक उचित उपवर्ग है। हम थ्रेशोल्ड ग्राफ के इंजेक्टिव क्रोमैटिक इंडेक्स की गणना के लिए रैखिक समय एल्गोरिथ्म का भी प्रस्ताव करते हैं। एक ग्राफ G को एक पूर्ण EIC-ग्राफ के रूप में जाना जाता है यदि $\chi'_i(G)$, G के अधिकतम समूह में किनारों की संख्या के बराबर है। हम साबित करते हैं कि थ्रेशोल्ड ग्राफ सही EIC-ग्राफ हैं।

ग्राफ पर परिभाषित विभिन्न ऑपरेशन नए ग्राफ बनाने और विशेष ग्राफ वर्गों को संरचनात्मक रूप से चिह्नित करने के लिए एक महत्वपूर्ण विधि बनाते हैं। इस थीसिस में, हमने कुछ मानक ग्राफ संचालन और उत्पादों के तहत ग्राफ रंग भरने की इन विविधताओं का भी अध्ययन किया है, जिसमें कार्टेशियन उत्पाद, मजबूत उत्पाद, लेक्सिकोग्राफिक उत्पाद, कोरोना उत्पाद, एज कोरोना उत्पाद, जॉइन, उपखंड और एक ग्राफ के माइसिलस्क्रियन शामिल हैं।

Contents

Certificate	i
Acknowledgements	iii
Abstract	v
List of Figures	xv
List of Symbols	xxi
1 Introduction	1
1.1 Preliminaries	2
1.1.1 Graph-theoretic definitions and notations	2
1.1.2 Algorithmic preliminaries	4
1.2 Variations of graph coloring studied in the thesis	6
1.2.1 Neighbor-sum-2-distinguishing edge-weighting	7
1.2.2 Injective coloring	10
1.2.3 Exact square coloring	11
1.2.4 Injective edge coloring	14
1.2.5 Coloring of graph operations and products	15
1.3 Graph classes studied in the thesis	19
1.3.1 Bipartite graph	19
1.3.2 Chordal graph	22
1.4 Contributions of the thesis	26
1.5 Organization of the thesis	29

2	Neighbor-sum-2-distinguishing edge-weighting	33
2.1	Introduction	33
2.2	$\chi_{\Sigma>1}(G)$ of perfect elimination bipartite graphs	34
2.3	Neighbor-sum-2-distinguishing $\{1, 3\}$ -edge-weighting of graphs	39
2.3.1	Neighbor-sum-2-distinguishing $\{1, 3\}$ -edge-weighting of bipartite graphs	39
2.3.2	1-3-5 Conjecture for central graphs of general graphs	44
2.4	Minimum sum count of neighbor-sum-2-distinguishing $\{1, 3\}$ -edge-weighting of graphs	47
2.4.1	$\gamma_{\Sigma>1,\{1,3\}}(G)$ of bipartite graphs	48
2.4.2	Some observations of the minimum sum count of neighbor-sum-2-distinguishing edge-weighting of graphs	55
2.4.3	Central graphs of bipartite graphs	57
2.4.4	Central graphs of cycles and paths	65
2.4.5	Central graphs of split graphs	74
3	Injective coloring	77
3.1	Introduction	77
3.2	Injective coloring in bipartite graphs	78
3.2.1	NP-completeness proof for perfect elimination bipartite graphs	78
3.2.2	NP-completeness proof for star-convex bipartite graphs	81
3.2.3	NP-completeness proof for comb-convex bipartite graphs	83
3.2.4	Inapproximability of injective coloring in perfect elimination bipartite graphs	86
3.2.5	Injective coloring in biconvex bipartite graphs	88
3.2.6	Injective coloring in convex bipartite graphs	90
3.2.7	Injective coloring in chordal bipartite graphs	93
3.3	Injective coloring in chordal graphs	99
3.3.1	Injective coloring in proper interval graphs	100

3.3.2	Injective coloring in threshold graphs	108
3.3.3	Injective coloring in $K_{1,t}$ -free split graphs	109
4	Exact square coloring	117
4.1	Introduction	117
4.2	Exact square coloring in chordal graphs	118
4.2.1	NP-completeness proof for undirected path graph	118
4.2.2	Exact square coloring in proper interval graphs	121
4.2.3	Exact square coloring in threshold graphs	127
4.2.4	Exact square coloring in block graphs	130
4.3	Exact square coloring in bipartite graphs	146
4.4	Exact square clique number of graphs	146
4.4.1	NP-completeness proof for undirected path graphs	147
4.4.2	NP-completeness proof for perfect elimination bipartite graphs	148
4.4.3	NP-completeness proof for star-convex bipartite graphs	150
4.4.4	NP-completeness proof for comb-convex bipartite graphs	151
4.4.5	Exact square clique in proper interval graphs	152
4.4.6	Exact square clique in threshold graphs	153
4.4.7	Exact square clique in block graphs	154
4.4.8	Exact square clique in convex bipartite graphs	155
5	Injective Edge Coloring	157
5.1	Introduction	157
5.2	NP-completeness results	158
5.2.1	Perfect elimination bipartite graph	158
5.2.2	Star-convex bipartite graph	161
5.3	Linear time algorithms	164
5.3.1	Chain graph	164
5.3.2	Trees	172
5.3.3	Threshold graph	184

6	Coloring of graph operations and products	189
6.1	Introduction	189
6.2	Neighbor-sum-2-distinguishing edge-weighting	190
6.2.1	Cartesian product of graphs	190
6.2.2	Lexicographic product of graphs	192
6.2.3	Corona product of graphs	194
6.2.4	Edge corona product of graphs	196
6.2.5	Mycielskian of a graph	197
6.3	Injective coloring	199
6.3.1	Corona product of graphs	199
6.3.2	Edge corona product of graphs	200
6.4	Exact square coloring of products of graphs	200
6.4.1	Cartesian product of graphs	201
6.4.2	Strong product of graphs	203
6.4.3	Lexicographic product of graphs	204
6.4.4	Corona product of graphs	205
6.4.5	Edge corona product of graphs	207
6.4.6	Join and subdivision	209
6.4.7	Mycielskian of a graph	211
6.5	Exact square clique number of products of graphs	214
6.5.1	Cartesian product of graphs	215
6.5.2	Strong product of graphs	216
6.5.3	Lexicographic product of graphs	217
6.5.4	Corona product of graphs	218
6.5.5	Edge corona product of graphs	218
6.5.6	Join and subdivision	219
6.5.7	Mycielskian of a graph	220
6.6	Injective chromatic index of graph products	221

6.6.1	Cartesian product of graphs	221
6.6.2	Lexicographic product of graphs	223
6.6.3	Corona product of graphs	225
6.6.4	Edge corona product of graphs	228
6.6.5	Join and subdivision	231
6.6.6	Mycielskian of a graph	232
7	Conclusion and Future Research	235
7.1	Contributions	235
7.2	Open Problems	238
	Bibliography	241
	Bio-Data	251

List of Figures

1.1	Neighbor-sum-distinguishing $\{1, 2\}$ -edge-weighting of G	8
1.2	Neighbor-sum-2-distinguishing $\{1, 3\}$ -edge-weighting of G	9
1.3	Proper coloring and injective coloring of $G = K_{1,8}$	11
1.4	The exact square coloring of G	12
1.5	(a) Graph G and (b) Injective edge coloring of G	15
1.6	(a) Cartesian product, $G \square H$, (b) strong product, $G \boxtimes H$, and (c) lexicographic product, $G \circ H$, of $G = P_5$ and $H = P_4$	17
1.7	(a) Corona product, $G \bullet H$, and (b) Edge corona product, $G \bullet^e H$, of $G = P_5$ and $H = P_4$	17
1.8	Mycielskian of $G = C_4$, $M(C_4)$	18
1.9	Central graph of C_4 , $C(C_4)$	19
1.10	A perfect elimination bipartite graph, G	20
1.11	A chain graph G	21
1.12	(a) A tree-convex bipartite graph, and (b) its associated tree $T = (X, E_X)$	22
1.13	Inclusion relationship between the subclasses of bipartite graphs.	23
1.14	A block graph G	24
1.15	A split graph G	24
1.16	A threshold graph G	25
1.17	A proper interval graph G	25
1.18	Inclusion relationship between subclasses of chordal graphs.	26

2.1	Gadgets I_1, I_2 and I_3	36
2.2	Construction of perfect elimination bipartite graph, H from G . The bold edges of H with red labels correspond to the edges included in the perfect edge elimination scheme of H	38
2.3	(a) Gadget H_5^3 . Each gadget H_{2i+1}^3 , $3 \leq i \leq 8$ in (b)-(g) is constructed by attaching gadgets H_{2i-1}^3 , H_1 or H_2 to p_{2i+1}^3 of H_{2i+1}^3 . The gadgets H_{2j+1}^1 , $j = 5, 7$ in (h) and (i) are constructed by attaching gadgets H_{2j+3}^3 , H_1 and H_2 to p_{2j+1}^1 of H_{2j+1}^1	42
2.4	Gadgets I_1 (left) and I_1' (right).	50
2.5	Gadget I_C	51
2.6	Variable gadget.	54
2.7	Modified Gadget I_C	56
2.8	$\{1, 3\}$ -edge-weighting of complete graphs with 1, 2 and 3 pairs of vertices having same incident sums, where the edges in black and red are assigned weight 1 and 3, respectively.	59
2.9	$\{1, 3\}$ -edge-weightings of complete graphs with 4 and 5 pairs of vertices having same incident sums obtained by Construction 1, where the edges in black and red are assigned weight 1 and 3, respectively.	61
2.10	Using Construction 2, we can extend a $\{1, 3\}$ -edge-weighting from K_8 to K_{11} with 4 pairs of vertices having same incident sums, namely, $\sigma(v_1) = \sigma(v_2) = 14$, $\sigma(v_3) = \sigma(v_4) = 16$, $\sigma(v_5) = \sigma(v_6) = 18$ and $\sigma(v_7) = \sigma(v_8) = 20$. Here, the black and red edges correspond to the edges of weight 1 and 3, respectively, and the incident sum of a vertex at each step of the construction is marked in blue.	63
2.11	Neighbor-sum-2-distinguishing $\{1, 3\}$ -edge-weighting of central graphs of cycles with $n \leq 8$. Here, black, red and green edges represent edges of weight 1, 3 and 5, respectively, and $\sigma(v)$ is highlighted in blue for each vertex.	66

2.12	Neighbor-sum-2-distinguishing $\{1, 3\}$ -edge-weighting of central graphs of cycles of lower order. Note that each white vertex obtained by subdividing the edges of the cycle is assigned incident sum 2.	68
2.13	Partition of the vertices of a cycle into levels such that each level contains 8 vertices, except the first and last levels that contain v_1 and either 6, 8, 10 or 12 vertices, respectively	70
2.14	Examples of the central graphs of cycles, where the edges marked red correspond to those edges whose weight is updated to 3 by Algorithm 2 for the cases of H_{15}, H_{17}, H_{19} and H_{21} ($ L^0 = 6, 8, 10, 12$ respectively). The value of $\varepsilon_3(v)$ for each vertex is highlighted in green.	73
3.1	Construction of perfect elimination bipartite graph, H from G	79
3.2	Construction of star-convex bipartite graph, H from G	82
3.3	Construction of comb-convex bipartite graph, H from G	85
3.4	Injective coloring of a biconvex bipartite graph obtained from Algorithm 3.	90
3.5	The algorithm cannot use more than Δ colors.	93
3.6	Injective coloring of a convex bipartite graph obtained from Algorithm 2.	94
3.7	Injective coloring of a Type-2 proper interval graph obtained from Algorithm 7.	105
3.8	Example of a graph with $\Delta = 3$	106
3.9	Injective coloring of a threshold graph G	109
4.1	The construction of H from G and its clique tree.	120
4.2	An example of a proper interval graph G and H , obtained from Algorithm 8, where H is a bipartite graph.	127
4.3	(a) An example of a proper interval graph G . (b) The obtained graph H is not a bipartite graph. (c) The exact square coloring of G obtained from Algorithm 8.	127

4.4	A block graph and its refined cut-tree.	133
4.5	An example of a Type-1 block graph.	134
4.6	An example of a Type-2 block graph.	135
4.7	The construction of a perfect elimination bipartite graph H from G . . .	149
5.1	The construction of perfect elimination bipartite graph H from G . . .	160
5.2	The construction of star-convex bipartite graph H from G	162
5.3	An example of a Type-I chain graph with red edges denoting the edges of matching saturating X . The optimal injective edge coloring of G , obtained from Algorithm 11, is shown in the right.	166
5.4	An example of a Type-II chain graph and its maximum neighbor graph $M_X(G)$, where the bold lines denote the edges of $M_X(G)$, the red dotted lines denote the edges of matching M_P and the number on the vertices of T_X and T_Y denote the label f assigned to these vertices, used by Algorithm 11 to obtain injective edge coloring ω . . .	173
5.5	Example of a caterpillar graph and its optimal injective edge coloring using 2 colors obtained from Algorithm 13.	179
5.6	Example of a caterpillar graph and its optimal injective edge coloring using 3 colors obtained from Algorithm 13.	179
5.7	Example of a tree and its optimal injective edge coloring using 2 colors obtained from Algorithm 13.	181
5.8	Example of a tree and its optimal injective edge coloring using 3 colors obtained from Algorithm 13.	182
5.9	Examples of two threshold graphs and their optimal injective edge coloring using $m^*(C)$ and $m^*(C) + C $ colors, respectively, obtained from Algorithm 15.	187
6.1	The exact square coloring of $C_6 \square P_3$	202
6.2	The exact square coloring of $P_3 \boxtimes P_4$	204

6.3	The exact square coloring of corona product of C_4 and H	207
6.4	The exact square coloring of $C_4 \cdot^e K_3$	209
6.5	The graph $B_{M(G)}$ obtained from $G = K_{1,3}$ and the exact square coloring of $M(G)$	214
6.6	A proper vertex coloring and injective edge coloring of $G = C_4$ and $H = P_3$, respectively, are given on the left, and an optimal injective edge coloring $\omega_{C_4 \square H}$ of $C_4 \square P_3$ is given on the right.	223
6.7	A proper vertex coloring and injective edge coloring of C_4 and H , respectively, are given on the left, and an optimal injective edge coloring $\omega_{C_4 \cdot H}$ of $C_4 \cdot H$ is given on the right.	225
6.8	A proper vertex coloring and injective edge coloring of $G = P_3$ and $H = C_3$, respectively, are given on the left, and an optimal injective edge coloring $\omega_{P_3 \cdot^e C_3}$ of $P_3 \cdot^e C_3$ is given on the right.	230

List of Symbols

Symbol	Meaning
\mathbb{N}	Set of <i>natural numbers</i>
\mathbb{Z}	Set of <i>integers</i>
$\forall x$	For <i>all</i> x
$x \in X$	x is a <i>member</i> of X
$A \subseteq X$	A is a <i>subset</i> of X
$ X $	The <i>cardinality</i> of a set X
$A \cap B$	The <i>intersection</i> of A and B
$A \cup B$	The <i>union</i> of A and B
$A \setminus B$	The <i>set difference</i> of A and B
\emptyset	The <i>empty</i> set
\square	The <i>end of a proof</i>
	such that
$E(G)$	The set of <i>edges</i> of G
$V(G)$	The set of <i>vertices</i> of G
$d(v)$ or $d_G(v)$	The <i>degree</i> of the vertex v
Δ or $\Delta(G)$	The <i>maximum degree</i> in G
δ or $\delta(G)$	The <i>minimum degree</i> in G

$N(v)$ or $N_G(v)$	neighborhood of v in G
$N[v]$ or $N_G[v]$	closed neighborhood of v in G
$N_G^2(v)$	The set of vertices which are at distance 2 from the vertex v in G
$d(u, v)$	The <i>distance</i> between u and v in G
$G[H]$	The subgraph of G <i>induced</i> by the vertices of H
$G = (X, Y, E)$	The <i>bipartite graph</i> G with vertex set $X \cup Y$
\overline{G}	The <i>complement</i> of an undirected graph G
$\omega(G)$	The <i>clique number</i> of G
$\theta(G)$	The <i>clique cover number</i> of G
$\alpha(G)$	The <i>independence number</i> of G
$\lambda_{p,q}(G)$	The $L(p, q)$ - <i>labeling number</i> of G
K_n	The <i>complete graph</i> on n vertices
C_n	The <i>cycle</i> on n vertices
P_n	The <i>path</i> on n vertices
$K_{m,n}$	The <i>complete bipartite graph</i> on $m + n$ vertices partitioned into an m -independent set and an n -independent set
$\chi_\Sigma(G)$	The least value of k such that G has a neighbor-sum-distinguishing edge-weighting from $\{1, 2, \dots, k\}$
$\chi_{\Sigma>1}(G)$	The least value of k such that G has a neighbor-sum-2-distinguishing edge-weighting from $\{1, 2, \dots, k\}$
$\gamma_W(G)$	The minimum number of distinct sums produced by a neighbor-sum- distinguishing W -edge-weighting of G
$\gamma_{\Sigma>1, \{1,3\}}(G)$	The minimum number of distinct sums produced by a neighbor-sum- 2-distinguishing $\{1, 3\}$ -edge-weighting of G
$\chi(G)$	The <i>chromatic number</i> of G

$\chi_i(G)$	The <i>injective chromatic number</i> of G
$\chi^{[\#2]}(G)$	The <i>exact square chromatic number</i> of G
$\omega^{[\#2]}(G)$	The <i>exact square clique number</i> of G
$\chi'_i(G)$	The <i>injective chromatic index</i> of G
$\chi'(G)$	The <i>chromatic index</i> of G
$G \square H$	The <i>Cartesian product</i> of two graphs G and H
$G \boxtimes H$	The <i>strong product</i> of two graphs G and H
$G \circ H$	The <i>lexicographic product</i> of two graphs G and H
$G \bullet H$	The <i>corona product</i> of two graphs G and H
$G \bullet^e H$	The <i>edge corona product</i> of two graphs G and H
$G + H$	The <i>join</i> of two graphs G and H
$S(G)$	The <i>subdivision</i> of a graph G
$M(G)$	The <i>Mycielskian</i> of a graph G
$C(G)$	The <i>central graph</i> of a graph G
P	The class of <i>deterministic</i> polynomial-time solvable problems
NP	The class of <i>nondeterministic</i> polynomial-time solvable problems
ZPP	The class of <i>zero-error probabilistic polynomial</i> problems

