DYNAMIC BEHAVIOUR OF INSTRUMENTED SINGLE VERTICAL AND BATTER PILES AND PILE GROUPS UNDER VERTICAL AND COUPLED VIBRATIONS: FIELD MODEL TESTING AND THEORETICAL STUDY

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by

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CERTIFICATE

This is to certify that the thesis entitled "**Dynamic Behaviour of Instrumented Single Vertical and Batter Piles and Pile Groups under Vertical and Coupled Vibrations: Field Model Testing and Theoretical Study**", submitted by **Mr. Rohit Ralli** to the Indian Institute of Technology Delhi, is a record of bonafide research work carried out by him under my supervision and is worthy of consideration for the award of **Doctor of Philosophy** degree. The research reported in this thesis has not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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ABSTRACT

In the present study, the response of a single vertical pile, *P1* ($\beta = 0^o$) and single batter piles, *P2* ($\beta = 10^o$) and *P3* ($\beta = 20^o$) and 2 x 2 vertical pile group (VPG) and batter pile group (BPG) for different dynamic force level under vertical and coupled vibrations are investigated. All the piles in BPG are unilaterally inclined at an angle of 20^o with vertical. The centre to centre spacing between the piles are kept at a distance of 3d for both the pile groups. The dynamic test results are presented in terms of the frequency versus amplitude response, frequency versus contact pressure response, distribution of dynamic bending moments and axial loads along the pile in vertical and batter piles and pile groups. Comparative assessment of test results of both the vertical and batter piles with respect to the peak resonant frequency and resonant amplitudes, pile soil separation length, dynamic axial loads and bending moments under vertical and coupled vibrations has been emphasized. The effect of loading direction i.e., coupled vibration in along and across the inclination of batter pile are also compared in a comprehensive manner for single batter piles and pile groups.

To investigate the complex nonlinear characteristics of the soil-pile system for single vertical and batter piles and pile groups under machined induced vertical and coupled (horizontal and rocking) vibrations, the field dynamic test results are compared with the theoretical results. Two soil models are used to study the linear and nonlinear behaviour of the vertical and batter piles under dynamic loading. Nonlinear responses of the piles are predicted by introducing the weak boundary zone parameters and soil-pile separation length for different excitation forces. The variation of frequency dependent stiffness and damping of single vertical and batter piles and pile groups are also investigated and compared for different excitation intensities under vertical and coupled vibrations using the theoretical approaches. The pile-soil-pile interaction phenomenon of the 2 × 2 vertical and batter pile group are also investigated by calculating the group efficiency ratios (GER) of the pile group for different excitation intensities and vibration modes.

Various design charts are proposed based on the finding of the study, to estimate the soil-pile separation lengths and boundary zone parameters for the single vertical and batter piles and 2 × 2 pile group with different pile batter angles under vertical and coupled vibrations. The applicability of the proposed design charts is validated for full-scale vertical and batter prototype piles using scaling law. It is found that the proposed design curves can be used to determine the nonlinear frequency-amplitude responses of the full-scale pile foundations under both vertical and coupled vibrations for similar soil-pile conditions. The charts are also validated using experimental data of various literature to check the reliability of the design charts. It is found that the proposed design charts can be used to determine the nonlinear frequency versus amplitude responses of the pile foundations under vertical and coupled vibrations for similar type of soil-pile conditions.

Keywords: Vertical pile, Batter pile, vertical vibration, coupled vibration, soil-pile separation, dynamic axial load, dynamic bending moment, linear approach, nonlinear approach, stiffness, damping, loading direction, GER.

सार

वर्तमान अध्ययन में, सिंगल वर्टिकल पाइल, *P*1 (β = 0°) और सिंगल बैटर पाइल्स, *P*2 (β = 10°) और *P*3 (β = 20°) और 2 x 2 वर्टिकल पाइल ग्रुप (VPG) और बैटर पाइल की प्रतिक्रिया ऊर्ध्वाधर और युग्मित कंपनों के तहत विभिन्न गतिशील बल स्तर के लिए समूह (बीपीजी) की जांच की जाती है। बीपीजी में सभी ढेर ऊर्ध्वाधर के साथ 20° के कोण पर एकतरफा झुके हुए हैं। ढेर के बीच केंद्र से केंद्र की दूरी दोनों ढेर समूहों के लिए 3 डी की दूरी पर रखी जाती है। गतिशील परीक्षण के परिणाम आवृत्ति बनाम आयाम प्रतिक्रिया, आवृत्ति बनाम संपर्क दबाव प्रतिक्रिया, गतिशील झुकने वाले क्षणों के वितरण और ऊर्ध्वाधर और बैटर ढेर और ढेर समूहों में ढेर के साथ अक्षीय भार के संदर्भ में प्रस्तुत किए जाते हैं। शिखर गुंजयमान आवृत्ति और गुंजयमान आयामों, ढेर मिट्टी की जुदाई लंबाई, गतिशील अक्षीय भार और ऊर्ध्वाधर और युग्मित कंपन के तहत झुकने वाले क्षणों के संबंध में ऊर्ध्वाधर और बल्लेबाज पाइल दोनों के परीक्षण परिणामों के तुलनात्मक मूल्यांकन पर जोर दिया गया है। लोडिंग दिशा के प्रभाव यानी, बैटर पाइल के झुकाव के साथ-साथ युग्मित कंपन की तुलना सिंगल बैटर पाइल्स और पाइल समूहों के लिए व्यापक तरीके से की जाती है।

मशीनी प्रेरित ऊर्ध्वाधर और युग्मित (क्षैतिज और रॉकिंग) कंपन के तहत एकल ऊर्ध्वाधर और बल्लेबाज ढेर और ढेर समूहों के लिए मिट्टी-ढेर प्रणाली की जटिल गैर-रेखीय विशेषताओं की जांच करने के लिए, सैद्धांतिक परिणामों के साथ क्षेत्र गतिशील परीक्षण परिणामों की तुलना की जाती है। डायनेमिक लोडिंग के तहत वर्टिकल और बैटर पाइल्स के लीनियर और नॉनलाइनियर व्यवहार का अध्ययन करने के लिए दो मृदा मॉडल का उपयोग किया जाता है। विभिन्न उत्तेजना बलों के लिए कमजोर सीमा क्षेत्र मापदंडों और मिट्टी-ढेर पृथक्करण लंबाई को शुरू करके बवासीर की गैर-

रेखीय प्रतिक्रियाओं की भविष्यवाणी की जाती है। एकल ऊर्ध्वाधर और बल्लेबाज ढेर और ढेर समूहों की आवृत्ति निर्भर कठोरता और नमी की भिन्नता की भी जांच की जाती है और सैद्धांतिक दृष्टिकोण का उपयोग करके लंबवत और युग्मित कंपन के तहत विभिन्न उत्तेजना तीव्रता के लिए तुलना की जाती है। विभिन्न उत्तेजना तीव्रता और कंपन मोड के लिए ढेर समूह के समूह दक्षता अनुपात (जीईआर) की गणना करके 2 × 2 लंबवत और बल्लेबाज ढेर समूह की ढेर-मिट्टी-ढेर बातचीत घटना की भी जांच की जाती है।

सिंगल वर्टिकल और बैटर पाइल्स के लिए मिट्टी-पाइल पृथक्करण लंबाई और सीमा क्षेत्र मापदंडों का अनुमान लगाने के लिए अध्ययन की खोज के आधार पर विभिन्न डिज़ाइन चार्ट प्रस्तावित किए गए हैं और ऊर्ध्वाधर और युग्मित कंपन के तहत विभिन्न पाइल बैटर कोणों के साथ 2 × 2 पाइल समूह हैं। प्रस्तावित डिजाइन चार्ट की प्रयोज्यता स्केलिंग कानून का उपयोग करके पूर्ण पैमाने पर लंबवत और बैटर प्रोटोटाइप ढेर के लिए मान्य है। यह पाया गया है कि प्रस्तावित डिजाइन घटता का उपयोग समान मिट्टी-ढेर स्थितियों के लिए ऊर्ध्वाधर और युग्मित कंपन दोनों के तहत पूर्ण पैमाने पर ढेर नींव की गैर-आवृत्ति आवृत्ति-आयाम प्रतिक्रियाओं को निर्धारित करने के लिए किया जा सकता है। डिजाइन चार्ट की विश्वसनीयता की जांच करने के लिए विभिन्न साहित्य के प्रयोगात्मक डेटा का उपयोग करके चार्ट को भी मान्य किया जाता है। यह पाया गया है कि प्रस्तावित डिजाइन चार्ट का उपयोग समान प्रकार की मिट्टी-ढेर स्थितियों के लिए ऊर्ध्वाधर और युग्मित कंपनों के तहत ढेर नींव की गैर-रेखीय आवृत्ति बनाम आयाम प्रतिक्रियाओं को निर्धारित करने के लिए किया जा सकता है।

कीवर्ड: वर्टिकल पाइल, बैटर पाइल, वर्टिकल वाइब्रेशन, कपल्ड वाइब्रेशन, सॉइल-पाइल सेपरेशन, डायनेमिक एक्सियल लोड, डायनेमिक बेंडिंग मोमेंट, लीनियर अप्रोच, नॉनलाइनियर अप्रोच, स्टिफनेस, डंपिंग, लोडिंग डायरेक्शन, जीईआर।

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LIST OF SYMBOLS

Most of the symbols are defined as they appear in the thesis. Some of the most common symbols, which are used repeatedly, listed below.

Roman symbols

a_o	Dimensionless frequency
$\overline{a_o}$	Dimensionless frequency for composite medium
A	Cross-sectional area of the pile
A_1	Area of the topmost element of the pile
A_H	Amplitude of horizontal motion
$A_{H1\text{-}res}$, $A_{H2\text{-}res}$	First and second resonant amplitudes for horizontal motion
A_s	Steady state amplitude
A_V	Amplitude of vertical motion
$A_{V\text{-}res}$	Resonant amplitude for vertical motion
b	Pile radius plus thickness of inner soil medium (= $R + t_m$)
c	Coefficient of pile internal damping
c_u^c	Horizontal damping of embedded footing or pile cap
c_{uu}^1	Horizontal damping constant of single pile
$c_{u\psi}^c$	Cross damping of embedded footing or pile cap
$c_{u\psi}^1 = c_{\psi u}^1$	Cross damping constant of single pile
c_w^c	Vertical damping of embedded footing or pile cap
c_{ww}^1	Vertical damping constant of single pile
c_ψ^c	Rocking damping of embedded footing or pile cap
$c_{\psi\psi}^1$	Rocking damping constant of single pile
C_{cu1}, C_{cu2}	Stiffness and damping parameters of half-space for horizontal vibration
C_{cw1}, C_{cw2}	Stiffness and damping parameters of half-space for vertical vibration

$C_{c\psi 1}$, $C_{c\psi 2}$	Stiffness and damping parameters of half-space for rocking vibration
C_{uu}	Total horizontal damping constants of pile
$C_{u\psi}$	Total cross damping constants of pile
$C_{\psi\psi}$	Total rocking damping constants of pile
d	Diameter of the pile
D	Damping ratio
D_i	Damping ratio corresponding to frequency of vibration ω_i
D_s	Dimensionless damping constant
D_{sm}	Dimensionless damping constant of inner medium
e	Eccentric distance of the rotating masses
E_b	Young's modulus of the bearing stratum
E_p	Young's modulus of the pile
E_s	Young's modulus of the soil along the pile
f_{c1}, f_{c2}	Dimensionless cross stiffness and damping parameters
f_{mc1}, f_{mc2}	Dimensionless cross stiffness and damping constants of the composite medium
f_{mu1}, f_{mu2}	Dimensionless horizontal stiffness and damping constants of the composite medium
f_{mw1}, f_{mw2}	Dimensionless vertical stiffness and damping constants of the composite medium
$f_{m\psi 1}$, $f_{m\psi 2}$	Dimensionless rocking stiffness and damping constants of the composite medium
f_n	Resonant frequency for vertical vibration
f_{n1}, f_{n2}	First and second resonant frequencies for coupled vibration
f_{u1}, f_{u2}	Dimensionless horizontal stiffness and damping parameters
f_{w1}, f_{w2}	Dimensionless vertical stiffness and damping parameters
$f_{\psi 1}$, $f_{\psi 2}$	Dimensionless rocking stiffness and damping parameters
f_{st}	Static flexibility of a single pile
$[f]$	Group flexibility matrix
$F(A)$	Restoring force
F_i	Dimensionless function

g	Acceleration due to gravity
G	Shear modulus of soil
$G^{'}$	Imaginary part of the complex shear modulus of soil
G^{*}	Complex shear modulus of outer soil medium
G_{cb}	Dynamic shear modulus of soil below the footing
G_{cs}	Dynamic shear modulus of the side layers of footing
G_{m}	Shear modulus of inner soil medium
G_{m}^{*}	Complex shear modulus of inner weak soil medium
h	Embedded depth of footing or pile cap
H	Height (length) of the element
$H_{1},\ H_{2}$	Horizontal end forces of element corresponding to end displacements
i	Complex number ($i=\sqrt{-1}$)
I	Moment of inertia of pile cross section
I_{1}	Moment of inertia of the topmost element of the pile
I_{ψ}	Mass moment of inertia about a horizontal axis passing through the center of gravity
k_{um}	Horizontal stiffness of the composite medium
k_{us}	Complex horizontal stiffness of soil per unit length of pile
k_{ut}	Complex horizontal stiffness of soil at the pile tip
k_{wm}	Vertical stiffness of the composite medium
k_{ws}	Complex vertical stiffness of soil per unit length of pile
k_{wt}	Complex vertical stiffness of soil at the pile tip
$k_{\psi m}$	Rocking stiffness of the composite medium
$k_{\psi s}$	Complex rocking stiffness of soil per unit length of pile
$k_{\psi t}$	Complex rocking stiffness of soil at the pile tip
$\bar{k}_{u}$	Horizontal static stiffness of single pile
$k_{u}^{(1)}$	The soil stiffness of the outer medium for horizontal vibration
$k_{u}^{(2)}$	The soil stiffness of the inner medium for horizontal vibration
k_{u}^{c}	Horizontal stiffness of embedded footing or pile cap

k_{uu}^{1}	Horizontal stiffness of single pile
$k_{u\psi}^{1} = k_{\psi u}^{1}$	Cross stiffness of single pile
$k_{u\psi}^{c}$	Cross stiffness of embedded footing or pile cap
$\bar{k}_{w}$	Vertical static stiffness of single pile
k_{w}^{c}	Vertical stiffness of embedded footing or pile cap
k_{ww}^{1}	Vertical stiffness of single pile
k_{ψ}^{c}	Rocking stiffness of embedded footing or pile cap
$k_{\psi\psi}^{1}$	Rocking stiffness of single pile
K_0	Modified Bessel functions of order zero
K_1	Modified Bessel functions of order one
K^G	Complex group stiffness
$K_e(A)$	Equivalent linear stiffness
K_{ij}	Elements of the structure matrix
K_{uu}	Total horizontal stiffness of pile
$K_{u\psi}$	Total cross stiffness of pile
K_{ww}	Total vertical stiffness of pile
$K_{\psi\psi}$	Total rocking stiffness of pile
K_{mu}^{1}	Total complex horizontal stiffness of single pile in composite medium
K_{mw}^{1}	Total complex vertical stiffness of single pile in composite medium
$K_{m\psi}^{1}$	Total complex rocking stiffness of single pile in composite medium
$K_{mu\psi}^{1} = K_{m\psi u}^{1}$	Total complex cross stiffness of single pile in composite medium
K_{uu}^{1}	Complete horizontal stiffness of single pile
$K_{u\psi}^{1} = K_{\psi u}^{1}$	Complete cross stiffness of single pile
K_{ww}^{1}	Complete vertical stiffness of single pile

$K_{\psi\psi}^{1}$	Complete rocking stiffness of single pile
K_{u}^{G}	Horizontal stiffness of pile group
$K_{u\psi}^{G}$	Cross stiffness of pile group
K_{w}^{G}	Vertical stiffness of pile group
K_{ψ}^{G}	Rocking stiffness of pile group
$\left[k_t\right]$	Element stiffness matrix of the soil underlying the pile tip
$\left[k_u\right]$	Element stiffness matrix for horizontal vibration
$\left[k_w\right]$	Element stiffness matrix for vertical vibration
$\left[K\right]$	Structure stiffness matrix
$\left[K_u\right]$	Structure stiffness matrix of single pile for horizontal vibration
$\left[K_w\right]$	Structure stiffness matrix of single pile for vertical vibration
l_s	Separation length between pile and soil
L	Length of the pile
m	Mass of eccentric rotating part in oscillator
m_{eff}	Effective mass of the system
m_s	Total mass of footing or pile cap
M	Real amplitude of moment
M_1, M_2	Moment of element corresponding to end rotations
$M(t)$	Total moment of excitation for coupled vibration
$M_e(t)$	Excitation moment
n	Number of piles in a group
N	SPT value
N_1, N_2	Vertical end forces of element corresponding to end displacements
N_{st}	Static axial force (load of the pile)
$N_x(t)$	Resultant horizontal reaction acting on embedded surface (sides) of footing
$N_\psi(t)$	Reactive moment of forces acting on footing sides about center of gravity

p	Lateral soil resistance
p_u	Ultimate soil resistance
$P1$	Single vertical pile with pile batter angle of 0 degrees
$P2$	Single batter pile with pile batter angle of 10 degrees
$P3$	Single batter pile with pile batter angle of 20 degrees
$P(t)$	Horizontal excitation force
$\{P\}$	Vector of forces
r_1	Radius of the topmost element of the pile
r_c	Radius of footing or pile cap
R	Radius of circular pile
R_{inter}	strength reduction factor
$R_x(t)$	Horizontal reaction at the footing base
$R_\psi(t)$	Reactive moment of forces acting at footing base about center of gravity
s	Spacing of piles in a group
s_u	Undrained shear strength
S_{cu1}, S_{cu2}	Stiffness and damping parameters of side layer for horizontal vibration
S_{cw1}, S_{cw2}	Stiffness and damping parameters of side layer for vertical vibration
$S_{c\psi1}$, $S_{c\psi2}$	Stiffness and damping parameters of side layer for rocking vibration
S_{ou1}, S_{ou2}	Horizontal stiffness and damping parameters of outer medium
S_{mu1}, S_{mu2}	Dimensionless stiffness and damping parameters of composite medium for horizontal vibration
S_{mw1}, S_{mw2}	Dimensionless stiffness and damping parameters of composite medium for vertical vibration
$S_{m\psi1}$, $S_{m\psi2}$	Dimensionless stiffness and damping parameters of composite medium for rocking
S_{u1}, S_{u2}	Real and imaginary parts of the dimensionless complex stiffness in horizontal direction (u)

S_{w1}, S_{w2}	Real and imaginary parts of the dimensionless complex stiffness in vertical direction (w)
$S_{\psi 1}$, $S_{\psi 2}$	Real and imaginary parts of the dimensionless complex stiffness in rotation or rocking (ψ)
t	Time
t_m	Thickness of inner soil medium
u	Horizontal displacement of pile
u_1, u_2	Horizontal end displacements of the element
VPG	Vertical pile group
BPG	Batter pile group
V_{La}	Lysmer's analog velocity
V_n	Shear wave velocity of soil in the lowest layer
V_s	Shear wave velocity of the soil
w	Vertical displacement of pile
w_1, w_2	Vertical end displacements of the element
w_{res}	Resonant amplitude of vertical vibration
W	Weight of eccentric rotating part in oscillator
W_s	Total weight of footing or total static load on pile
x_i , x_j	Distances of each pile from centre of gravity of foundation
y	Pile deflection
y_c	Critical pile deflection
z_c	Height of the center of gravity above the footing base
z_e	Height of horizontal excitation force above the center of gravity of footing

Greek symbols

α	Complex dynamic interaction factor
α_1	Real part of the complex dynamic interaction factor
α_2	Imaginary part of the complex dynamic interaction factor
β	Pile batter angle

α_{ij}	Dynamic interaction factor of pile group
α_{st}	Static interaction factor
$[\alpha]$	Matrix of interaction factors
$[\alpha]_H$	Interaction matrix for horizontal translation and rotations
$[\alpha]_V$	Interaction matrix for vertical translation
β	Frequency-independent soil material damping
δ	Loss angle
δ_r	Relative embedment
δ_{dij}	Dynamic displacement of pile *i* due to load on pile *j*
δ_{sij}	Static displacement of pile *j* due to its own load
$\{\delta\}$	Vector of displacements
θ	Angle of eccentricity
ϕ	Friction angle of soil
φ	Angle between the direction of load and the plane in which piles lie
ω	Circular frequency
ω_1, ω_2	The frequencies corresponding to the points of interaction between the response curve and a line passing through the origin of coordinates
ω_i	The components of the rotational vector
ω_m	Damped resonant circular frequency
ω_n	Undamped resonant circular frequency
ω_{n1}, ω_{n2}	Two resonant frequencies of coupled vibration
μ	Mass of the pile per unit length
ρ	Mass density
ν	Poisson's ratio of soil
ν_m	Poisson's ratio of inner zone
ν_s	Poisson's ratio of outer zone
λ, λ'	Real part and imaginary part of Lame's constant
ξ	Effective mass coefficient

ψ	Rotation (rocking) in vertical plane
ψ_1, ψ_2	Rotational (rocking) end displacements of the element
ψ_r	Amplitude of rocking motion
$\psi_{r1\text{-}res}$, $\psi_{r\,2\text{-}res}$	First and second resonant amplitudes for rocking motion
$\varepsilon^H_{2i,2j}$	Complex elements of $[\alpha]^{-1}_H$ that correspond to the moments associated with rotations
$\varepsilon^H_{2i-1,2j-1}$	Complex elements of $[\alpha]^{-1}_H$ that corresponds to the horizontal forces associated with horizontal displacements
ε^V_{ij}	Elements of $[\alpha]^{-1}_V$
Λ	Complex frequency parameter
Δ	Relative volume change
Δ_h	Maximum amplitude of pile for horizontal vibration
Δ_r	Maximum amplitude of pile for rocking vibration
Δ_v	Maximum amplitudes of pile for vertical motion
Ω	Backbone curve

Abbreviations

BEM	Boundary element method
BH	Bore hole
C.G.	Center of gravity
DC	Direct-current
Deg	Degree
DOF	Degree of freedom
Eq.	Equation
FE	Finite element
FEM	Finite element method
Fig.	Figure
GER	Group efficiency ratio
Hz	Hertz

IS	Indian standard
kN	Kilo Newton
L.L.	Liquid limit
m	Meter
mm	Millimeter
M.P.F.	Mass participation factor
P.L.	Plastic limit
Rad	Radian
RPM	Revolutions per minute
SDOF	Single degree of freedom
SPT	Standard penetration test
USCS	Unified soil classification system
2-D	Two dimensional
3-D	Three dimensional