50

Transient Analysis of Some Finite Simple Birth - Death Models

by

JAGANNATH DASS

Department of Mathematics

Submitted in fulfilment of the requirements of the degree of

DOCTOR OF PHILOSOPHY

INDIAN INSTITUTE OF TECHNOLOGY, DELHI

October, 1987

TO THE

MEMORY OF MY GRAND MOTHER

AND TO HER

SON AND DAUGHTER-IN-LAW

CERTIFICATE

It is certified that the thesis entitled "TRANSIENT ANALYSIS OF SOME FINITE SIMPLE BIRTH-DEATH MODELS" presented by Sh. JAGANNATH DASS is worthy of consideration for the award of the Degree of DOCTOR OF PHILOSOPHY and is a record of the original bonafide research work carried out by him under my guidance and supervision and that the results contained in it have not been submitted in part or in full to any other University or Institute for the award of any degree or diploma.

I certify that he has persued the prescribed course of research.

(Prof. O.P. SHARMA) 21.X.87
Supervisor
Department of Mathematics
Indian Institute of Technology
Hauz Khas, New Delhi-110016.

ACKNOWLEDGEMENTS

I wish to express my sincere gratitude and appreciation to Prof. O.P. Sharma who as my thesis supervisor has been a source of valuable inspiration and heartful encouragement. I would be indebted to him for long for his patient guidance and invaluable suggestions throughout the tenure of this research work. The large amount of time and painstaking efforts he timely devoted to my work are especially appreciated.

I am indeed grateful to Prof. M.M. Chawla, Head, Department of Mathematics and Prof. M.K. Jain, Deputy Director, IIT Delhi for their timely help and suggestions.

I owe a great deal to Prof. O.P. Bhutani, Prof. N.S. Kambo of the Department and Prof. J. Nanda, Dean, U.G. Studies, IIT Delhi, for their heartful advice and constant encouragement.

I am thankful to Dr. B.R. Handa, Dr. M.C. Puri, Dr. J.B.Srivastava and Dr. Suresh Chandra of the Department for their timely suggestions. My sincere thanks is also due to Dr.N.Ravichandran, I.I.M.,Ahmedabad.

My sincere thanks are also due to all my fellow research scholars and friends who have selflessly helped provide me the inspirable atmosphere throughout my stay at IIT Delhi.

I shall certainly fail in my duty if I don't express my deep sense of gratitude to my younger brother Sh. Raghunath Dass, S.R.A., Department of Civil Engineering, IIT Delhi for his constant help in preparation of the thesis.

I gratefully acknowledge the financial support and conducive research facilities I have received from IIT Delhi for the completion of this research work.

Personally, I owe a lot to my family members for their untiring patience and devotion.

Last but not least, a word of sincere thanks is also due to Ms Neelam Dhody for her neat and efficient typing of the thesis and to Ms Bahl and Mrs. Khurana for their elderly affection.

Jagannath Dass

(JAGANNATH DASS)

SYNOPSIS

The study of Stochastic Models of queueing, machine interference and reliability systems plays a pivotal role in the real life problems as these models probe the service systems prone to congestion. The initial state of such a system (empty or non-empty) affects the transient study of the system whereas the steady-state case remains independent of the initial conditions. But steady state analysis is inappropriate in certain applied situations since the time horizon of such a operation naturally terminates or the steady-state measures of system performance become redundant. Furthermore, we also find that there is a physical constraint on the source or capacity of the system.

Thus, in order to have a complete and meaningful knowledge about such a system we need investigate it with regards to

i) transient behaviour,

ii) arbitrary initial conditions,

iii) finite source/waiting room capacity.

But it is observed that the solutions incorporating above points are very difficult to obtain as the mathematics involved becomes very cumbersome and complicated and it is, therefore, seen that the solutions reported in the literature are usually obtained by relaxing some or all of the above considerations. However, in this thesis a new computable matrix approach is developed to study the closed form solutions of the aforesaid systems with regard to the above three concepts. In certain cases the busy period analysis has also been carried out with very encouraging results. The efficacy of the method developed is also illustrated by means of some worked out examples.

This dissertation has been divided into six chapters. Brief outline of each of the chapters is given below.

CHAPTER-I: The chapter entitled "Introduction" has been devoted to the survey of existing literature of the models under study. It also contains a concise resume of the results of recent developmentsin the said models and the highlights of some basic concepts connected with the models.

CHAPTER-II: In this chapter, closed form solution of transient behaviour of M/M/2/N queueing models with heterogeneous service rates at the two channels is obtained through the matrix approach.

It has been assumed in this model that the inter-arrival time and service time for 1st and 2nd server are negative exponentially distributed with parameter λ^{-1}, μ_1^{-1} and μ_2^{-1} respectively. Without loss of generality $\mu_1 > \mu_2$ is assumed which also implies that first arriving unit in the queue (when it is empty) joins the 1st server for service and thereafter the arriving unit goes to the counter which it finds free. The maximum number of customers is restricted to N. Furthermore, there are arbitrary initial 'i' ($0 \le i \le N$) number of customers waiting at time zero. $P_n(t)$ being the probability that there are n customers in the system at time t, a neat closed form solution is obtained as

$$P_n(t) = \begin{cases} \mu_1 B + e^{-(\lambda+\mu_1+\mu_2)t} \sum_{j=1}^{n} \{(1-\delta_{io})(-1)^i \frac{\mu_1}{\lambda} \rho^{(i+1)/2} A_{ij} \\ \qquad + \delta_{io} A_{oj}\} e^{-\alpha_{Nj} t\sqrt{c}}, \quad n = 0 \\ (\mu_1+\mu_2) B\rho^n + e^{-(\lambda+\mu_1+\mu_2)t} \sum_{j=1}^{n} (-1)^{n-i} \rho^{(n-i)/2} A_{ij} h_n(\alpha_{Nj}) \\ \qquad \times e^{-\alpha_{Nj} t\sqrt{c}}, \quad n = 1,2,\ldots,i \\ (\mu_1+\mu_2) B\rho^n + e^{-(\lambda+\mu_1+\mu_2)t} \sum_{j=1}^{N} (-1)^{n-i} \rho^{(n-i)/2} A_{nj}\, h_n(\alpha_{Nj} \\ \qquad \times e^{-\alpha_{Nj} t\sqrt{c}}, \quad n = i+1, i+2, \ldots, N \end{cases}$$

where $\rho = \lambda/(\mu_1+\mu_2)$, $B = (1-\rho)\{\mu_1(1-\rho) + \lambda(1-\rho^N)\}^{-1}$ and

A_{nj} $(n = 0,1,2,\ldots,N)$ have been suitably defined as the functions of α_{Nj} (the distinct eigenvalues of the coefficient matrix of system), λ, μ_1 and μ_2. After suitable assumptions, steady-state results obtained by taking $t \to \infty$ agree with the results available in the literature.

The results for M/M/1/N can be derived as a particular case by putting $\mu_2 = 0$. Also by putting $\mu_1 = \mu_2$ we can get the results for M/M/2/N queueing system having equal service rates for both the servers.

The third section of the chapter is devoted to the derivation of initial busy period distributions generated by (1) 1st server and (2) both the servers. With proper assumptions, result obtained there from have been verified with the results available in the literature on busy period analysis of simple queueing system.

<u>CHAPTER-III</u>: This chapter centres around the transient behaviour of M/M/c/N queueing system. The closed form transient expression for the state probabilities of this model taking arbitrary 'i' $(0 \le i \le N)$ number of customers present at time zero is obtained. With $\rho = \lambda/c\mu$, the system probabilities $P_n(t)$ are given by

$$P_n(t) = \pi_n + e^{-(\lambda+c\mu)t} \sum_{k=1}^{N} A_{nk}\, e^{-\alpha_{Nk} t\sqrt{\lambda c\mu}} \qquad \rho \neq 1$$

where A_{nk} $(n = 0,1,2,\ldots,N)$ are the functions of λ,μ,c and $\alpha_{n,k}$. The distinct roots α_{Nk} $(k = 1,2,\ldots,N)$ are the eigen values of the coefficient matrix of the system and π_n are the equilibrium probabilities given by

$$\pi_n = \frac{(c\mu)^n}{n} \pi_o \;, \quad 0 \le n \le c$$

$$= \frac{c^c}{c!} \rho^n \pi_o, \quad c \le n \le N$$

with

$$\pi_o = [\sum_{k=1}^{c-1} \frac{(c\rho)^k}{k!} + \frac{c^c}{c!} (\rho^c - \rho^{N+1})(1-\rho)^{-1}]^{-1}$$

By taking $c = 1,2$ and putting $i = 0$ the results available in the literature can be derived as particular cases from this model.

In the second section of the chapter, initial busy period distributions generated by k servers $(k = 1,2,\ldots,c)$ have been worked out. With suitable assumptions, the important parameters obtained from these are shown to agree with the known results.

CHAPTER-IV: In this chapter, the initial busy period analysis and transient solutions of Markovian birth-death machine interference system with r operatives is carried out.

The probabilistic approach to machine interference model is known to have started with the work of Aschroft in which he had studied the steady-state solution of M/G/1 machine interference model. However, so far no explicit closed form solution is available in the literature for finding transient analysis and initial busy period distributions for such model.

We consider the problem of N identical and automatic machines under the care of r-operatives $(1 \leq r \leq N)$ where the failure rate of each machine and repair time of each repairman are negative exponentially distributed with parameter λ^{-1} and μ^{-1} respectively. We also assume that there is an arbitrary number i $(0 \leq i \leq N)$ of machine breakdowns before the start of repairing.

We have $b_j(t)$ and $P(n,t)$ as the initial busy period p.d.f. generated by $j(j = 1,2,\ldots,r)$ operatives and transient probabilities of the system respectively and obtain

$$b_j(t) = \frac{d}{dt} P(j-1,t)$$

and

$$P(n,t) = \pi_n + e^{-(N\lambda+r\mu)t} \sum_{k=1}^{N} \pi_{nk}\, e^{-\alpha_{Nk} t}$$

where π_{nk} depend upon λ,μ,r and α_{Nk} and α_{Nk} $(k = 1,2,\ldots N)$

are the real distinct eigenvalues of the coefficient matrix of the system.

Taking $\rho = \lambda/\mu$ we obtain the steady-state solutions

$$\pi_n = \binom{N}{n} \rho^n \pi_0, \quad 0 < n < c$$

$$= \binom{N}{n} \frac{n!}{c^{n-c}\, c!} \rho^n \pi_0, \quad c \le n \le N$$

$$\pi_0 = \left[\sum_{n=0}^{c-1} \binom{M}{n} \rho^n + \sum_{n=c}^{N} \binom{N}{n} \frac{n!}{c^{n-c}\, c!} \rho^n \right]^{-1}$$

which are well known results.

Putting $r = 1$ and $i = 1$, we get the busy period generated by 1st operative and it agrees with recent work by Bunday.

CHAPTER-V: In this chapter, we discuss a limited space multiserver Markovian queueing model with balking and reneging but without passing.

The objective of the 2nd section of the chapter is to find the transient effects of customer's impatience upon the development of waiting lines of the M/M/c/N queueing system.

The third section of the chapter investigates the queueing situation where customers leave the system in the same order in which they arrive due to the physical restriction that no passing is allowed. The concept of "without passing" propounded by Alan Washburn has been emphasised in this section. Here, balking probability is taken as

$$B_n = (1-n/N), \quad n = 0,1,2,\ldots,N$$

which indicates that N is a measure of customer willingness to join the queue. A customer reneges after joining the queue if it decides that the certain length of waiting-time will be larger than to be tolerated. This time is taken as a random variable with density function

$$R_{n-r}(t) = (n-r)\alpha e^{-(n-r)\alpha t}$$

and reneging function

$$R_f = (n-r)\alpha$$

where r is the number of servers in the multiserver queue and μ and λ being the usual service and arrival rates respectively. It is also assumed that there are $i(0 \le i \le r)$ arbitrary initial number of customers present. Considering the above assumptions, the transient probabilities come out as

$$P_n(t) = A_n + \sum_{j=1}^{N} A_{nj}\, e^{-\alpha_{N,j} t}$$

and the constants have been suitably defined.

Steady state results are found to be

$$A_n = \binom{N}{n} \left(\frac{\delta}{\gamma} \right)^n A_o, \quad 0 \le n \le i$$

$$= N!\ \delta^n\ \Gamma(r\gamma)/\{(N-n)!\,(r-1)!\,\gamma^{r-1}\,\Gamma(r\gamma+n-r)\}\, A_o, \quad i+1 \le n \le N$$

with

$$A_o = \sum_{n=0}^{r-1} \binom{N}{n} \left(\frac{\delta}{\gamma} \right)^n + \sum_{n=r}^{N} N!\ \beta^n\ \Gamma r\gamma/\{(N-n)!\,(r-1)!\,\gamma^{r-1}\,\Gamma(r\gamma+n-r)\}$$

where $\gamma = \mu/\alpha$ and $\delta = \lambda'/\alpha$.

Putting $i = 0$ and $r = 1$ one gets the steady state solution available for single server queueing problem with balking and reneging. Further, assuming the reneging factor $\alpha = 0$ one can also get the transient solution of Markovian machine interference model with r operatives having arrival rate $\lambda' = \lambda/N$ discussed in Chapter-IV.

<u>CHAPTER-VI</u>: Extensive research is reported in the area of stochastic analysis of redundant repairable. reliability systems. The successful analysis of each system depends on the complexity of the underlying stochastic process. In the literature most of the analysis reported so far confines to the measures like stationary availability in steady state case. However, in many reliability evaluation models the transient behaviour of the system is important and may even be essential. The purpose of this chapter is to provide the transient analysis of a general Markovian birth-death

reliability system.

$\{X(t), t \geq 0\}$ is considered as an integer valued birth-death stochastic process. We take λ_i as the failure rate when 'i' units are operable and μ as the repair rate of the system. It is assumed that $P_{ij}(t)$ be the probability of the system in state j at time t given that the system is in state i $(0 \leq i \leq n)$ at $t = 0$ and $q_j(t)$ be the probability of the system in state j at time 't' given the information that state zero has not been initiated in $(0,t)$. Further, setting $i = n$ (n is the total number of units) the required transient probabilities are obtained as

$$P_{nj}(t) \approx P_j(t) = a_j + \sum_{i=1}^{n} a_{ij}\, e^{-\alpha_{n,i}\, t}$$

and

$$q_j(t) = \sum_{i=1}^{n} b_{ij}\, e^{-\beta_{n,i}\, t}$$

where a_{ij} and b_{ij} are the functions of the roots $\alpha_{n,i}$ and $\beta_{n,i}$ respectively which are the distinct sets of eigen values of the coefficient matrix of the system under different conditions.

The steady-state solutions are obtained as

$$a_j = \frac{1}{\mu^{n-j}} \prod_{k=j+1}^{n} \lambda_k\, a_n\,, \quad 0 \leq j < n$$

with

$$a_n = \left[1 + \sum_{k=0}^{n-1} \frac{1}{\mu^{n-k}} \prod_{i=k+1}^{n} \lambda_i \right]^{-1}$$

which agree with the results available in the literature.

The solution procedure is presented for a general case and results are obtained for the case of cold and warm standby systems as well as parallel redundancy. The method is illustrated for special case $n = 2$ which is also applicable for $n \geq 2$. It has been verified for the case $n = 3$.

CONTENTS