ON THE DEGENERATE WHITTAKER SPACE FOR $\mathrm{GL}_4(\mathfrak{o}_2)$

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On the degenerate Whittaker space for $\mathrm{GL}_4(\mathfrak{o}_2)$

by

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To
My Parents

Certificate

This is to certify that the thesis entitled **On the degenerate Whittaker space for** $\mathrm{GL}_4(\mathfrak{o}_2)$ submitted by **Ms. Ankita Parashar** to the **Indian Institute of Technology Delhi**, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi
July 2025

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Abstract

Let $\mathfrak{o}_2$ be a finite principal ideal local ring of length two. Let $N = \left\{ \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} : X \in M_n(\mathfrak{o}_2) \right\}$. Since $N \cong M_n(\mathfrak{o}_2)$, define a character ψ of N by $\psi(X) := \psi_0(\mathrm{tr}(X))$, where ψ_0 is a primitive character of $\mathfrak{o}_2$. For any representation π of $\mathrm{GL}_{2n}(\mathfrak{o}_2)$, the degenerate Whittaker space $\pi_{N,\psi}$ of π is defined to be the maximal subspace of π on which N acts by ψ. Then $\pi_{N,\psi}$ is a representation of $\mathrm{GL}_n(\mathfrak{o}_2)$. Our work provides an explicit description of $\pi_{N,\psi}$ for certain representations π of $\mathrm{GL}_4(\mathfrak{o}_2)$. Mainly, we focus on two classes of representations which are strongly cuspidal representations and certain parabolically induced representations.

For strongly cuspidal representations π, we give an explicit description of $\pi_{N,\psi}$. Alongwith, we verify a special case of a conjecture of Dipendra Prasad. Moreover, we demonstrate that the representation $\pi_{N,\psi}$ of $\mathrm{GL}_2(\mathfrak{o}_2)$ is multiplicity-free. Our proof employs an explicit construction of strongly cuspidal representations along with the application of Mackey theory. In the context of parabolically induced representations π, we consider those induced from the subgroups denoted by P and Q of $\mathrm{GL}_4(\mathfrak{o}_2)$ and explicitly describe $\pi_{N,\psi}$. In particular, for representations induced from Q, we prove that the associated degenerate Whittaker space $\pi_{N,\psi}$ is also multiplicity-free.

Our work aligns with one of works of Dipendra Prasad on degenerate Whittaker space over finite fields and it extends the understanding of degenerate Whittaker spaces over finite principal ideal local rings.

सार

मान लीजिए $\mathfrak{o}_2$ लंबाई दो का एक परिमित मुख्य आदर्श स्थानीय वलय है। मान लीजिए $N = \left\{ \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} : X \in M_n(\mathfrak{o}_2) \right\}$। चूँकि $N \cong M_n(\mathfrak{o}_2)$, N का एक वर्ण ψ इस प्रकार परिभाषित कीजिए: $\psi(X) := \psi_0(\mathrm{tr}(X))$, जहाँ ψ_0, $\mathfrak{o}_2$ का एक आदिम वर्ण है। $\mathrm{GL}_{2n}(\mathfrak{o}_2)$ के किसी भी निरूपण π के लिए, π का अपभ्रंश व्हिटेकर समष्टि $\pi_{N,\psi}$ को π का अधिकतम उपसमष्टि माना जाता है जिस पर N, ψ द्वारा कार्य करता है। तब $\pi_{N,\psi}$, $\mathrm{GL}_n(\mathfrak{o}_2)$ का प्रतिनिधित्व है। इस कार्य में, हम $\mathrm{GL}_4(\mathfrak{o}_2)$ के कुछ जटिल निरूपणों π के लिए $\pi_{N,\psi}$ का स्पष्ट रूप से वर्णन करते हैं।

हमारा ध्यान दो महत्वपूर्ण श्रेणियों के निरूपणों $\mathrm{GL}_4(\mathfrak{o}_2)$ के लिए $\mathrm{GL}_2(\mathfrak{o}_2)$ के निरूपण के रूप में π के विवरण को समझने पर है। ये दृढ़तापूर्वक कस्पिडल निरूपण और कुछ परवलयिक रूप से प्रेरित निरूपण हैं। दृढ़तापूर्वक कस्पिडल निरूपण π के लिए $\pi_{N,\psi}$ के विवरण में, हम प्रसाद के एक अनुमान के एक विशेष मामले की पुष्टि करते हैं। इसके अलावा, हम यह भी साबित करते हैं कि $\pi_{N,\psi}$ एक बहुलता मुक्त निरूपण है। परवलयिक रूप से प्रेरित निरूपणों में, हम $\mathrm{GL}_4(\mathfrak{o}_2)$ के दो उपसमूहों, P और Q से प्रेरित निरूपणों के लिए पतित व्हिटेकर स्पेस की जांच करते हैं जो $\mathrm{GL}_4(\mathbb{F}_q)$ के अधिकतम परवलयिक उपसमूहों के अनुरूप के रूप में कार्य करते हैं। Q से प्रेरित प्रतिनिधित्व के लिए, संबंधित पतित व्हिटेकर स्पेस $\pi_{N,\psi}$ बहुलता–मुक्त है।

हमारा कार्य, परिमित क्षेत्रों पर पतित व्हिटेकर स्पेस पर दीपेन्द्र प्रसाद के कार्यों में से एक के साथ संरेखित है और यह परिमित प्रमुख आदर्श स्थानीय वलयों पर पतित व्हिटेकर स्पेस की समझ को बढ़ाता है।

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