

ON TRAVELLING SALESMAN PROBLEM FOR DYNAMICAL SYSTEMS

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ON TRAVELLING SALESMAN PROBLEM FOR DYNAMICAL SYSTEMS

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CERTIFICATE

This is to certify that the thesis entitled “**On Travelling Salesman Problem for Dynamical Systems**”, submitted by **Eswara Venkata Kumar Dhulipala** to the Indian Institute of Technology Delhi, for the award of the degree of **Doctor of Philosophy** in Electrical Engineering, is a record of the bonafide research work carried out by him under my supervision and guidance. The thesis has met the standards, fulfilling the requirements of the regulations relating to the award of the degree.

The results contained in this thesis have not been submitted, either in part or in full, to any other University or Institute for the award of any degree or diploma, to the best of my knowledge.

Prof. Deepak Patil
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ABSTRACT

Travelling Salesman Problem (TSP) involves a salesman visiting a given set of points by starting and terminating at the same point. TSP requires that the salesperson visit each point only once while optimizing the required cost functions, such as time, fuel, fare, etc. In applications such as aerial surveillance, an aircraft acting as the salesman can not change direction instantaneously and have *inherent dynamics restricting the motion*. Also, in applications such as atomic force microscopy raster scanning, the tip of an Atomic Force Microscope (AFM) visits a set of locations on a sample to construct an image of the sample. In raster scanning, the tip of an AFM visits a set of locations by moving from left to right and right to left in a top-down approach. When moving from left to right or right to left, the *dynamics of AFM are approximated with a second-order Linear Time Invariant (LTI) system*. Similarly, when moving from top to bottom, the *dynamics of AFM are approximated with a second-order LTI system*. This thesis considers the TSP assuming the salesman is a dynamical system with inherent dynamics restricting the motion. In particular, we consider Dubins Vehicle, i.e., a system with only forward motion and constant speed, as a salesman in TSP visiting the set of points given in the two-dimensional Euclidean space. The Dubins vehicle as a salesman in TSP is termed the Dubins Travelling Salesman Problem (DTSP). Similarly, we consider another variant of TSP with dynamics in which the second-order LTI system acts as a salesman visiting the set of points in the phase plane, unlike DTSP, in which points are in 2D Euclidean space. We call the TSP for an LTI system the Linear Time Invariant Travelling Salesman Problem (LTITSP).

Owing to the computational complexity involved in TSP, there is *no polynomial time complexity algorithm that provides an optimal solution to the TSP*. For the TSP with system dynamics, the complexity is twofold as compared to the TSP since an additional system's motion constraint is added. TSP has motivated the researchers to opt for approximation algorithms that provide a trade-off between time complexity and solution quality. The addition of system dynamics to TSP increases the complexity of the problem, requiring new and better algorithms that balance time complexity and solution quality.

In this thesis, as a first step towards addressing DTSP, a *worst case upper bound is obtained on DTSP tour length assuming DTSP tour sequence is the same as Euclidean Travelling Salesman Problem (ETSP) tour sequence*. It is noted that, in the worst case, *any algorithm that uses an ETSP tour sequence* is a constant factor approximation algorithm for DTSP. Next, to address DTSP, an algorithm based on dynamic programming called Modified Dynamic Programming Algorithm (MDPA) is presented. MDPA computes the tour sequence and orientation

angles simultaneously. We show that *tour length by MDPA for DTSP is constant times the optimal ETSP tour length*. Next, we present an algorithm called Angle Bisector Algorithm (ABA) to address DTSP. ABA assumes that the DTSP tour sequence is the same as the ETSP tour sequence and computes the orientation angle at each point. ABA works on the principle of angle bisector. We show that the *ABA is a constant factor approximation algorithm when the Euclidean distance between any two points in the given set of points is at least 4ρ where ρ is the minimum turning radius*. To improve ABA, we present a modified version of ABA called Modified Angle Bisector Algorithm (M-ABA). M-ABA also assumes that the DTSP tour sequence is the same as the ETSP tour sequence. However, M-ABA considers more than one orientation angle at each point, resulting in a well-known problem in the literature called Generalized Travelling Salesman Problem (GTSP) or One in a Set TSP. We use *Noon-Bean transformation to convert GTSP to Asymmetric Travelling Salesman Problem (ATSP)* and solve ATSP using the well-established techniques from the literature. The presented approaches are validated through numerical simulations.

In the literature, one of the approaches to addressing DTSP considers a fixed sequence of points in a subset of a given set of points. For a k fixed sequence of points in the given n set of points, the orientation angle at each of these k points is computed so that the path length through this fixed sequence of k points is minimum. In the literature, this problem is termed k -Point Dubins Problem (k PDP). To address DTSP, next, another fixed sequence of k points starting with the second point in the previous sequence along with another point from the unvisited points is considered. This process is repeated till all the points are visited. In this thesis, to address k PDP, an *algorithm called Elimination Algorithm (EA) based on the Groebner basis is presented*. The presented algorithm is validated for the case of $k = 3$ through numerical simulations.

Finally, in this thesis, we consider an LTITSP that is inspired by DTSP. In addressing LTITSP, we first preset a class of points called Class of Optimal Tour (COT) points, for which it is easy to find the time-optimal tour sequence. Using the COT points, we present an algorithm called the Combining Time Optimal Tours (CTOT) algorithm to address LTITSP. The presented algorithm is demonstrated for a particular case of an LTI system, i.e., a double integrator system. For the double integrator case, a *lower bound on tour time is presented, assuming the points are generated according to uniform probability* in the phase plane. Also, we assume that the number of points tends to infinity. The presented approach is validated through numerical simulations.

Keywords: Travelling Salesman Problem, Dubins Travelling Salesman Problem, Generalized Travelling Salesman Problem, Noon-Bean Transformation, Dubins Vehicle, Angle Bisector, Dynamic Programming, Groebner Basis, k -Point Dubins Problem, Approximation Algorithms.

सारांश

ट्रैवलिंग सेल्समैन समस्या (TSP) में एक सेल्समैन को एक निर्धारित बिंदुओं के समूह का भ्रमण करना होता है, जिसकी शुरुआत और समाप्ति एक ही बिंदु पर होती है। इस समस्या में यह आवश्यक होता है कि सेल्समैन प्रत्येक बिंदु को केवल एक बार भ्रमण करे, और समय, ईंधन, किराया आदि जैसी लागत फलनों का अनुकूलन किया जाए।

ऐसे अनुप्रयोगों में जैसे कि एरियल सर्विलांस, एक विमान सेल्समैन के रूप में कार्य करता है जो तुरंत दिशा नहीं बदल सकता और जिसकी गति पर आंतरिक सीमाएं (डायनेमिक्स) लागू होती हैं। इसी प्रकार, एटॉमिक फोर्स माइक्रोस्कोप (AFM) रास्टर स्कैनिंग के अनुप्रयोग में, एटॉमिक फोर्स माइक्रोस्कोप (AFM) की टिप किसी सैंपल की छवि निर्माण के लिए उस पर स्थित बिंदुओं का भ्रमण करती है। रास्टर स्कैनिंग में, AFM की टिप ऊपर से नीचे की दिशा में बाएं से दाएं और दाएं से बाएं गति करती है। जब AFM बाएं से दाएं या दाएं से बाएं गति करता है, तब उसकी गतिशीलता को द्वितीय-क्रमीय लीनियर टाइम इनवैरिएंट (LTI) सिस्टम से निरूपित किया जाता है। इसी तरह, जब AFM ऊपर से नीचे गति करता है, तब भी उसकी गतिशीलता को द्वितीय-क्रमीय LTI सिस्टम से मॉडल किया जाता है।

यह शोध-प्रबंध TSP का अध्ययन इस परिकल्पना के तहत करता है कि सेल्समैन एक गतिशील प्रणाली (dynamical system) है जिसकी गति पर आंतरिक सीमाएं लागू हैं। विशेष रूप से, हम ड्युबिन्स व्हीकल (Dubins Vehicle) पर विचार करते हैं — अर्थात् एक ऐसी प्रणाली जो केवल आगे की दिशा में स्थिर गति से चल सकती है — जो द्वि-आयामी यूक्लिडियन स्पेस में बिंदुओं का भ्रमण करती है। इस स्थिति में TSP को ड्युबिन्स ट्रैवलिंग सेल्समैन समस्या (DTSP) कहा जाता है।

इसी प्रकार, हम TSP का एक और रूप मानते हैं जिसमें सेल्समैन के रूप में एक द्वितीय-क्रमीय LTI सिस्टम कार्य करता है और बिंदु फेज प्लेन में होते हैं, न कि दो-आयामी यूक्लिडियन स्थान में। इसे हम लीनियर टाइम इनवैरिएंट ट्रैवलिंग सेल्समैन समस्या (LTITSP) कहते हैं।

TSP की संगणनात्मक जटिलता के कारण, इसका इष्टतम समाधान प्राप्त करने के लिए कोई बहुपद-समय (polynomial-time) एल्गोरिथ्म उपलब्ध नहीं है। जब सिस्टम डायनेमिक्स को जोड़ा जाता है, तो समस्या की जटिलता और अधिक बढ़ जाती है, जिससे समाधान और भी कठिन हो जाता है। इस कारण से शोधकर्ता ऐसे अनुमानात्मक एल्गोरिथ्म (approximation algorithms) का उपयोग करते हैं जो समय जटिलता और समाधान गुणवत्ता के बीच संतुलन बनाए रखते हैं।

इस शोध में, DTSP की दिशा में पहला कदम यह है कि यदि DTSP दूर क्रम वही हो जो Euclidean TSP (ETSP) में हो, तो DTSP दूर की अधिकतम लंबाई की एक ऊपरी सीमा (worst-case upper bound) प्राप्त की जाती है। यह दिखाया गया है कि इस स्थिति में कोई भी एल्गोरिथ्म DTSP के लिए एक नियत गुणांक (constant factor) अनुमानात्मक एल्गोरिथ्म होता है।

इसके बाद, Modified Dynamic Programming Algorithm (MDPA) नामक एक डायनामिक प्रोग्रामिंग आधारित एल्गोरिथ्म प्रस्तुत किया गया है, जो DTSP दूर क्रम और उन्मुखीकरण कोणों (orientation angles) को एक साथ गणना करता है। यह दर्शाया गया है कि MDPA द्वारा प्राप्त दूर की लंबाई, ETSP की इष्टतम लंबाई के एक नियत गुणांक के बराबर होती है।

DTSP को हल करने के लिए Angle Bisector Algorithm (ABA) नामक एक और एल्गोरिथ्म प्रस्तुत किया

गया है, जो यह मानता है कि DTSP का टूर क्रम ETSP के समान है और प्रत्येक बिंदु पर उन्मुखीकरण कोण की गणना करता है। ABA कोण द्विभाजक सिद्धांत पर कार्य करता है। यह सिद्ध किया गया है कि जब बिंदुओं के बीच की न्यूनतम यूक्लिडियन दूरी कम से कम 4ρ हो (जहाँ ρ न्यूनतम टर्निंग त्रिज्या है), तो ABA एक नियत गुणांक अनुमानात्मक एल्गोरिथ्म होता है।

ABA को बेहतर बनाने हेतु Modified Angle Bisector Algorithm (M-ABA) प्रस्तुत किया गया है, जो ABA के समान टूर क्रम मानकर चलता है, परंतु प्रत्येक बिंदु पर एक से अधिक उन्मुखीकरण कोणों को विचार में लेता है। इसके कारण यह समस्या Generalized Travelling Salesman Problem (GTSP) बन जाती है। हम Noon-Bean transformation का उपयोग कर GTSP को Asymmetric TSP (ATSP) में परिवर्तित करते हैं और इसे साहित्य में उपलब्ध तकनीकों द्वारा हल करते हैं। इन विधियों को संख्यात्मक सिमुलेशनों द्वारा सत्यापित किया गया है।

साहित्य में, DTSP को हल करने के लिए एक और दृष्टिकोण यह है कि किसी दिए गए बिंदुओं के समूह में से k निश्चित बिंदुओं की अनुक्रम तय कर ली जाए और फिर प्रत्येक पर उन्मुखीकरण कोण इस प्रकार चुना जाए कि कुल मार्ग की लंबाई न्यूनतम हो। इस समस्या को k -Point Dubins Problem (k PDP) कहा जाता है। इस शोध में k PDP को हल करने हेतु Elimination Algorithm (EA) प्रस्तुत किया गया है, जो ग्राएबनर बेसिस पर आधारित है। इसे $k = 3$ के लिए संख्यात्मक परीक्षण द्वारा सत्यापित किया गया है।

अंततः, इस शोध में LTITSP पर विचार किया गया है, जो DTSP से प्रेरित है। LTITSP को हल करने के लिए, पहले एक विशेष प्रकार के बिंदुओं की श्रेणी प्रस्तुत की गई है जिसे Class of Optimal Tour (COT) कहा गया है, जिनके लिए समय-इष्टतम टूर क्रम निकालना सरल होता है। इन COT बिंदुओं का उपयोग कर हम Combining Time Optimal Tours (CTOT) नामक एल्गोरिथ्म प्रस्तुत करते हैं। इसे एक विशेष LTI प्रणाली — डबल इंटीग्रेटर सिस्टम (Double Integrator System) — के लिए प्रदर्शित किया गया है। डबल इंटीग्रेटर के मामले में, फेज प्लेन में बिंदुओं को समान संभावना वितरण से उत्पन्न मानकर, जब बिंदुओं की संख्या अनंत की ओर बढ़ती है, तब न्यूनतम टूर समय की एक सीमा प्रस्तुत की गई है। यह विधि भी संख्यात्मक सिमुलेशनों द्वारा प्रमाणित की गई है।

कीवर्ड्स: ट्रैवलिंग सेल्समैन समस्या, ड्युबिन्स ट्रैवलिंग सेल्समैन समस्या, सामान्यीकृत ट्रैवलिंग सेल्समैन समस्या, नून-बीन्स ट्रांसफॉर्मेशन, ड्युबिन्स क्लिकल, कोण द्विभाजक, डायनामिक प्रोग्रामिंग, ग्राएबनर बेसिस, k -पॉइंट ड्युबिन्स समस्या, अनुमानात्मक एल्गोरिथ्म।

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List of Abbreviations

LTITSP Linear Time Invariant Travelling Salesman Problem

DITSP Double Integrator Travelling Salesman Problem

DTSP Dubins Travelling Salesman Problem

TSP Travelling Salesman Problem

STSP Symmetric Travelling Salesman Problem

ATSP Asymmetric Travelling Salesman Problem

AFM Atomic Force Microscope

LTI Linear Time Invariant

ETSP Euclidean Travelling Salesman Problem

MDPA Modified Dynamic Programming Algorithm

DP Dynamic Programming

ABA Angle Bisector Algorithm

M-ABA Modified Angle Bisector Algorithm

GTSP Generalized Travelling Salesman Problem

k PDP k -Point Dubins Problem

3PDP 3-Point Dubins Problem

4PDP 4-Point Dubins Problem

EA Elimination Algorithm

CTOT Combining Time Optimal Tours

SVA Single-Vehicle Algorithm

AA Alternating Algorithm

RBTA Recursive Bead Tailing Algorithm

PSO Particle Swarm Optimisation

MAA Mean Angle Algorithm

LKH Lin–Kernighan-Helsgaun

OHA Optimized Heading Algorithm

List of Symbols

$\mathbb{R}$	One-dimensional space of real numbers
$\mathbb{R}^m$	m-dimensional space of real numbers
acos	$\arccos$
asin	$\arcsin$
atan	$\arctan 2$
ρ	Minimum turning radius of Dubins vehicle
L	Represents the left turn of a Dubins vehicle
OPT_E	Optimal tour length of ETSP
R	Represents the right turn of a Dubins vehicle
S	Represents the straight path of a Dubins vehicle