STABILITY ANALYSIS OF BIPED ROBOTS

ALINJAR DAN

DEPARTMENT OF MECHANICAL ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY DELHI

APRIL 2024

© Indian Institute of Technology Delhi (IITD), New Delhi, 2024

Stability Analysis of Biped Robots

by

Alinjar Dan

Department of Mechanical Engineering

Submitted

in fulfillment of the requirements of the degree of

Doctor of Philosophy

to the

INDIAN INSTITUTE OF TECHNOLOGY DELHI

April 2024

Certificate

This is to certify that the thesis entitled **"Stability Analysis of Biped Robots"**, submitted by **Mr. Alinjar Dan** to the Indian Institute of Technology Delhi, for the award of the degree of **Doctor of Philosophy**, is a record of the original, bona fide research work carried out by him under our supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations related to the award of the degree.

The results contained in this thesis have not been submitted in part or in full to any other University or Institute for the award of any degree or diploma to the best of our knowledge.

Prof. S.K. Saha
Department of Mechanical Engineering,
Indian Institute of Technology Delhi.

Dr. Rama Krishna K
Department of Mechanical Engineering,
Indian Institute of Technology Delhi.

Acknowledgements

I am grateful to everyone who was part of my journey in the last five and a half years.

I want to start by thanking the Ministry of Human Resource and Development (MHRD), Government of India, for providing the Prime Minister Research Fellowship (PMRF). I thank my supervisors, Prof. S.K. Saha and Rama Krishna K., for their continuous guidance. I will always be grateful to Prof. S.K.Saha, especially for his consistent encouragement to improve the thesis by carrying out more significant results and effectively communicating them through the thesis chapters. I have learnt the basics of writing a manuscript, improving it by adding suitable figures and highlighting the contributions from the long discussions with Prof. Rama Krishna K. I am especially thankful for his time and patience.

I want to thank my Student Research Committee members, Professor S. V. Modak, J.P. Khatait, and S. Bhasin, for their regular insightful feedback on my work. Their feedback has helped me to organize and improve my work.

I would like to thank my seniors, batch-mates and juniors at IIT Delhi for their constant support and encouragement. I start with seniors Prof. Suril Shah, Dr. Paramanand Nandihal, Dr. Riby Abraham. All of them graduated before or just after I started my PhD journey, but I have always been a phone call or a message away from their help. The same applies to other seniors Dr. Sasanka Sekhar Sinha, Dr. Anil Sharma, Dr. Rajeevlochana G C, and Dr. Suraj Bhat, who graduated in the last four years. Regarding batch-mates, I have a special mention for Dr. Rajesh Kumar, who was my go-to guy from the first day at IIT Delhi for each academic and administrative detail. The other seniors, batch-mates and juniors who have remained friends, philosophers and guides during my journey are Mr. Anil Patidar, Mr. Apurba Das, Mr. Udayan Banerjee, Mr. Nagendra Kumar, Mr. Ravi Tripathi, Mr. Surya Pratap Rathour, Mr. Hedayatullah, Mr. Chayan Paul, Mr. Neetish Patel, Mr. Sandeep Kumar, Mr. Rahul Sharma. Among them, I had the most discussions (work-related or otherwise) with Mr. Udayan Banerjee, for which I shall always be grateful. I especially thank Mr. Anil Patidar for lending his learnings to me, Mr. Chayan Paul for agreeing to do collaborative work when needed, and everyone else for being there when I needed them. I also had a nice friendship with some M.Tech and MSR guys in the last five years. The special mentions are Mr. Dinesh Kumar, Mr. Jay Patel, Mr. Sandeep Gehani, Mr. Himangshu, Mr. Harshil Patil, Mr. Arpan Tiwari, Mr. Rajat Varshney, Mr. Shahswat Jain, Mr. Bhanu Vardhan. I have learnt a lot from them as well.

I had a special opportunity to mentor a few students from other institutes nationwide as interns. I want to thank Prof. Saha for giving me this wonderful opportunity. The students I have mentored are Gurmann Singh Bhullar from G H Patel College of Engineering and Technology, Ms. Saniya Patwardhan from IIT Gandhinagar, Mr. Amit Sawant from VNIT, Mr. Hurath R from NIT Trichy, Ms. Garima Bansal from IIT Kharagpur. Also, as a PMRF, I mentored B.Tech final-year students at the Department of Mechanical and Automation Engineering of G.B. PANT DSEU OKHLA-I. The students were Aryan Choudhary, Milind Prakasha, Shaunkhoneil Chakraborty, and Md Yusuf, and my supervisor there was Dr. Arindam Chanda. I have learnt a lot from the process of mentoring and the mentees as well. I wish them all the very best in their future endeavours.

Now, I would like to thank my family members. Starting with my late father Mr. Aloke Dan, to whom I shall always be grateful for the upbringing he has given me. My mother, Mrs. Dalia Dan (Dutta), and my younger sister Ms. Alipsa Dan, have always been a pillar of strength during all the good, bad and ugly times. I am also thankful to all my other family members and friends from school life and college life (during graduation) who were in constant touch during this journey.

I dedicate my thesis to the scientific community all around the world. Hopefully, this thesis can make a change in the world, no matter how tiny it is in the larger scheme of things.

Alinjar Dan

© 2024, *Indian Institute of Technology Delhi*

Abstract

This thesis primarily focuses on the stability of biped robots under different circumstances. In the last six decades of biped research, the goal has been to make them capable enough to live with humans in the real world and be used in applications, namely, disaster management and extra-terrestrial locomotion, where the employment of human life is risky. To achieve this goal, the stability of bipeds in different environmental conditions needs to be investigated. While there have been many advancements in biped research, the limitations of surrounding environmental conditions, e.g., the terrain geometry, stance scenario, and magnitude of external disturbance, still exist on stable locomotion and stance stability.

At the beginning of this thesis, the dynamic modelling of a biped was performed in the dual algebra framework. Apart from conventional equations of motion, this formulation yielded analytical expressions for reaction forces along joint axes that had not been explored before. The equations of motion were used to model stable underactuated locomotions on various geometrical surfaces. A methodology was proposed to achieve stable locomotion on various surfaces,i.e., level ground, 2^o and 4^o inclined grounds, and a circular surface with radius of curvature 8m by altering the controller gains.

Further, the stance stability of bipeds against external disturbances was analyzed. If the external disturbance does not cause significant changes in the system's states, stability can be achieved by proper Centre of Mass (CoM) or Centre of Gravity (CG) placements without using any controller. An exact theory was developed to study the static stability of a planar biped in Euclidean space. The contact geometries were considered generic, unlike the special cases of line and point contact geometries in the recent literature.

If the external disturbance causes significant changes in the system's states such that the CoM placement is not sufficient for stability, controller action is required. It is also known as push recovery. In the present work, a novel control and optimization methodology has been developed and implemented on a 12-DoF biped for push recovery from different stance scenarios, i.e., double leg stance on a level ground, double leg stance on two level grounds offset by a height and single leg stance. Using newly introduced terms, 'Effective Disturbance Ratio' (EDR), and 'Effective Disturbance to Support Ratio' (EDSR), it was shown that our proposed methodology performed better than the existing methodologies for push recovery across different contact conditions. A real prototype was developed to test the simulation results.

In conclusion, starting with dynamic modelling, the thesis analyses the stability of bipeds in various scenarios and external conditions. While the underactuated locomotions show the possibility of energy-efficient gait on various terrains, stance stability studies analyze the stability of bipeds against external disturbances with or without using controller actions depending on the magnitude of the disturbance. The expressions of reaction forces along joint axes in dynamic modelling and the feasible CoM locations for stance stability can also be utilized to design a biped suitable for stable locomotions.

Keywords: Biped, Dynamics, Stability, Locomotion, Stance, Contact, Push Recovery.

© 2024, Indian Institute of Technology Delhi

सार

यह थीसिस प्राथमिक रूप से विभिन्न परिस्थितियों में दो–पैर रोबोटों की स्थिरता पर ध्यान केंद्रित करती है। पिछले साढ़े छ: दशकों में दो–पैर रोबोटों अनुसंधान का मुख्य उद्देश्य उन्हें मनुष्यों के साथ वास करना, विपणन में उपयोग करने के लिए पर्याप्त रूप से सक्षम बनाने का रहा है| विशेष रूप से उन्हें उन परिस्थितयो मे कार्य के लिए सक्षम बनाना जहां मानव जीवन का उपयोग जोखिमपूर्ण है, जैसे की आपदा प्रबंधन और अतिरिक्त–पृथ्वीय चलन| इस लक्ष्य को प्राप्त करने के लिए, विभिन्न पर्यावरणीय स्थितियों में दो–पैरों रोबोटों की स्थिरता का अध्ययन किया जाना आवश्यक है। जबकि दो–पैर रोबोटों अनुसंधान में कई अग्रसरताएँ हुई हैं, लेकिन स्थिर चलन और स्थिति स्थिरता पर आसपास के पर्यावरणीय शर्तों की सीमाएँ, जैसे, भूमि की ज्यामिति, स्थिति परिदृश्य, और बाह्य बल का परिमाण, अभी भी मौजूद हैं।

इस थीसिस की शुरुआत में, दो–पैर की गतिकीय मॉडलिंग डुअल अलजेब्रा की ढांचे में की गई थी। सामान्य गति के समीकरणों के अलावा, इस सूत्रीकरण ने जॉइंट अक्ष के ऊपर संयोजन बल के लिए विश्लेषणात्मक संव्याख्याओं को पैदा किया जो पहले अन्वेषित नहीं किए गए थे। समीकरणों का उपयोग विभिन्न ज्यामितीय सतहों पर स्थिर कम क्रियान्वित लोकोमोशन की मॉडलिंग के लिए किया गया। एक विधि प्रस्तावित की गई थी जिसमे नियंत्रक गेन्स को बदलकर विभिन्न सतहों पर दो–पैर की स्थिर चलन को प्राप्त किया जा सकता है, जैसे कि, समतल भूमि, 2^{o} और 4^{o} को ढली हुई भूमि, और वक्ररी वर्तुला जिसका त्रिज्या 8m है।

आगे, दो–पैरों की स्थिति स्थिरता के लिए बाह्य बलों का विश्लेषण किया गया। यदि बाह्य बल सिस्टम की स्थितियों में कोई प्रमुख परिवर्तन नहीं लाता है, तो स्थिति को उचित सेंटर ऑफ़ मास्स (CoM) या सेंटर ऑफ़ ग्रेविटी (CG) के द्वारा बिना किसी नियंत्रक का उपयोग किए स्थिर किया जा सकता है। यूक्लिडीयन स्पेस में एक समतल दो–पैर की स्थिरता का स्थिरता का एक सटीक सिद्धांत विकसित किया गया। संपर्क ज्यामितियों को साधारण माना गया, जो पिछले अनुसंधान में देखे गए रेखा और बिंदु संपर्क ज्यामितियों के विशेष मामलों से अलग और अधिक सामान्य है ।

यदि बाह्य बल सिस्टम की स्थितियों में प्रमुख परिवर्तन लाता है जिससे कि स्थिति के लिए CoM स्थानिकता पर्याप्त नहीं होता, तो नियंत्रक कार्रवाई की आवश्यकता होती है। इसे 'पुश रिकवरी' भी कहा जाता है। वर्तमान कार्य में, एक नवीन नियंत्रण और अनुकूलन पद्धति का विकसित किया गया है और इसे एक 12–DoF दो–पैर पर प्रस्तुत किया गया है, विभिन्न स्थिति परिदृश्यों से पुश रिकवरी के लिए, अर्थात समतल भूमि पर डबल पैरों की स्थिति, एक समतल भूमि पर डबल पैरों की स्थिति, जिसमें एक समतल भूमि को एक ऊँचाई से अलग किया गया है, और एक एकल पैरों की स्थिति। नए परिभाषित शब्दों, 'इफेक्टिव डिस्टर्बेंस रेश्यो' (EDR) और 'इफेक्टिव डिस्टर्बेंस

एंड सपोर्ट रेश्यो' (EDSR) का उपयोग करके, दिखाया गया कि हमारी प्रस्तावित पद्धति विभिन्न संपर्क शर्तों पर पुश रिकवरी के लिए मौजूदा पद्धतियों से बेहतर प्रदर्शन करती है। एक वास्तविक प्रोटोटाइप विकसित किया गया था ताकि सिमुलेशन परिणामों का परीक्षण किया जा सके।

निष्कर्ष में, डायनामिक मॉडेलिंग से शुरुआत करके, यह थीसिस विभिन्न परिदृश्यों और बाह्य परिस्थितियों में दो–पैर रोबोटों की स्थिरता को विश्लेषण करती है। जबकि कम क्रियान्वित लोकोमोशन विभिन्न भूमिकाओं पर ऊर्जा कुशल चाल की संभावना दिखाता है, स्थिति स्थिरता अध्ययन बाह्य बल के खिलाफ दो–पैर की स्थिरता का विश्लेषण करता है| बल के मात्रा के आधार पर निर्धारित किया जाता है की नियंत्रक क्रियाओं का प्रयोग चाहिए या नहीं। डायनामिक मॉडेलिंग से मिलता हुआ संयोजन बल और स्थिरता स्थिति में व्यक्तियों की व्यावहारिक CoM स्थानों का भी प्रयोग स्थिर लोकोमोशनों के लिए उपयोगी दो–पैर का डिज़ाइन करने के लिए किया जा सकता है।

© 2024, *Indian Institute of Technology Delhi*

Contents

© 2024, *Indian Institute of Technology Delhi*

© 2024, *Indian Institute of Technology Delhi*

List of Figures

© 2024, *Indian Institute of Technology Delhi*

© 2024, *Indian Institute of Technology Delhi*

© 2024, *Indian Institute of Technology Delhi*

© 2024, *Indian Institute of Technology Delhi*

List of Tables

ABBREVIATIONS

AI	Artificial Intelligence
AMPM	Angular Momentum Pendulum Moodel
CG	Centre of Gravity
CoM	Centre of Mass
CoP	Centre of Pressure
CP	Capture Point
CPG	Central Pattern Generator
CR	Capture Region
CTC	Computed Torque Control
DCBI	Dual Composite Body Inertia
DeNOC	Decoupled Natural Orthogonal Complement
DIM	Duality Inducing Matrix
DoF	Degrees of Freedom
EDR	Effective Disturbance Ratio
EDSR	Effective Disturbance to Stability Ratio
EL	Euler-Lagrange
FLC	Fuzzy Logic Control
GA	Genetic Algorithms
GA-FLC	Genetic Algorithms Fuzzy Logic Control
GA-NN	Genetic Algorithms Neural Network
GIM	Generalized Inertia Matrix
GRF	Ground reaction Force
HMCD	Human Motion Capture Data
IAV	Integrals of Absolute Value (IAV)
IC	Impedance Control
ICC	Instantaneous Centre of Curvature
IMU	Inertial Measurement Unit
IPM	Inverted Pendulum Model
IRC	Instantaneous Radius of Curvature
JARF	Joint Axis Reaction Force
LIPM	Linear Inverted Pendulum Model
MA	Mean of Amplitude

© 2024, *Indian Institute of Technology Delhi*

MCI	Matrix of Convective Inertia
MCIWO	Modified Chaotic Invasive Weed Optimization
MJM	Momentum Jacobian Matrix
MPC	Model Predictive Control
NE	Newton-Euler
NNC	Neural Network Control
NOC	Natural Orthogonal Complement
PD	Proportional Derivative
PID	Proportional-Integral-Derivative
PSO	Particle Swarm Optimization
QP	Quadratic Programming
RIM	Reaction Inertia Matrix
RL	Reinforcement Learning
RMCI	Reaction Matrix for Convective Inertia
RMS	Root Mean Square
RMPM	Reaction Mass Pendulum Model
RR	Revolute-Revolute
RVCI	Reaction Vector for Convective Inertia
SP	Support Polygon
SPA	Support Polygon Area
SVM	Support Vector Machine
URDF	Unified Robotics Description Format
VCI	Vector of Convective Inertia
VZMP	Virtual Zero Moment Point
VHP	Virtual Horizontal Plane
ZMP	Zero Moment Point
2-D	Two Dimensional
3-D	Three Dimensional

Notation

Here are some notations that are used in this thesis. A few guidelines that are used to categorise the notations are given below.

© 2024, *Indian Institute of Technology Delhi*

- Matrix symbols are upper-case letters in bold font.
- Vector symbols are lower-case letters in bold font.
- Points are upper-case letters in italic font.
- Scalar quantities and lines are lower-case letters in italic font.

Accents

$(\underline{\ \ })$	Dual vector
$(\dot{\ })$	First time-derivative
$(\ddot{\ })$	Second time-derivative
$(\hat{\ })$	Unit vector
$(\tilde{\ })$	Skew symmetric matrix form of a vector
$(\mid\mid)$	Magnitude of a vector
$(\ ^T\ )$	Transpose

Axes and Coordinates

$\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$	Three mutually perpendicular axes in globally fixed frame
$\boldsymbol{x}_i, \boldsymbol{y}_i, \boldsymbol{z}_i$	Three mutually perpendicular axes in body-fixed frame attached to i-th contact point
$(x; y)$	Coordinates of a point in $\boldsymbol{xy}$ frame (z coordinate $= 0$),
$(x_i; y_i)$	Coordinates of a point in $\boldsymbol{x}_i\boldsymbol{y}_i$ frame ($\boldsymbol{z}_i$ coordinate $= 0$),
$(x_h; y_h; z_h)$	Coordinates of hip joint in $\boldsymbol{xyz}$ frame
$(x_a; y_a; z_a)$	Coordinates of ankle joint in $\boldsymbol{xyz}$ frame

Greek Symbols

α_x, α_y, α_z	Angular acceleration along $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$-axis, respectively, rad/s^2
β_i	Angle formed by i-th link with $\boldsymbol{y}$-axis, rad (or deg)
$\dot{\beta}_i$	Rate of change of β_i, rad (or deg)/s
$\dot{\beta}_i^-$	Rate of change of β_i before impact, rad (or deg)/s
$\dot{\beta}_i^+$	Rate of change of β_i after impact, rad (or deg)/s

[8] Its own frame refers to the body-fixed frame whose $\boldsymbol{x}_i$-axis is along the link length, $\boldsymbol{z}_i$-axis is along the joint axis attached to it

© 2024, *Indian Institute of Technology Delhi*

$\Gamma(i)$	Parent link of i-th link
$\boldsymbol{\gamma}$	Joint torque vector due to gravity
$\overline{\boldsymbol{\gamma}}$	JARF due to gravity
γ_i	Torque at i-th joint due to gravity, N-m
ϵ	Dual coefficient, m^{-1}
ρ	Radius of curvature, m
ρ_A	IRC of point A, m
λ	Geometric kinematic nature of impressed motion, m
$\mathrm{Den}(\lambda)$	Denominator of ρ_A, unit less
λ_a, λ_b	Roots of $\mathrm{Den}(\lambda)$, m
$\Delta(\mathrm{Den})$	Discriminant of $\mathrm{Den}(\lambda)$, m^{-2}
$\Delta(t)$	Discriminant function to determine contact condition at time t, s
η	Viscous joint damping coefficient, N-m-s/rad
κ_m, κ_f	Curvature of m-curve and f-curve, m^{-1}
ι_i	Moment impulse at i-th joint of biped, N-m-s
μ	Static coefficient of friction between robot and ground, unit less
μ_i	Static coefficient of friction at i-th contact point (for multiple contacts) between robot and ground, unit less
υ_j	Ratio of normal contact force at j-th foot to weight of robot, unit less
ς	Small tolerance angular value, rad
$\boldsymbol{\omega}$	Angular velocity vector of a rigid body
$\boldsymbol{\omega}_i$	Angular velocity vector of i-th link
$\boldsymbol{\omega}_i^j$	Angular velocity vector of i-th link of j-th module
ω_x, ω_y, ω_z	Angular velocity along $\boldsymbol{x}$,$\boldsymbol{y}$, and $\boldsymbol{z}$-axis, respectively, rad/s
$\underline{\boldsymbol{\Omega}}$	Dual matrix to calculate MCI and RMCI
ψ, ζ	Coefficients forming characteristic equation of swing leg's GIM, rad (or deg) and s^{-1}
$\boldsymbol{\tau}$	Input joint torque vector for whole system
$\boldsymbol{\tau}^j$	Input joint torque vector for j-th module
$\overline{\boldsymbol{\tau}_p}^j$	Contribution by parent module on input joint torques of j-th module
$\boldsymbol{\tau}_{ctc}$	Input joint torque vector calculated by computed torque control
τ_i	Input torque at i-th joint , N-m
$\boldsymbol{\theta}$	Joint angle vector
$\dot{\boldsymbol{\theta}}$	Joint velocity vector
$\dot{\boldsymbol{\theta}}_r, \dot{\boldsymbol{\theta}}_l$	Joint velocity vector for right and left leg, respectively

© 2024, Indian Institute of Technology Delhi

θ_i	Angular displacement of i-th joint, rad
$\dot{\theta}_i$	Angular velocity of i-th joint, rad/s
Θ_i	Angle between $\boldsymbol{xyz}$ and $\boldsymbol{x}_i\boldsymbol{y}_i\boldsymbol{z}_i$ coordinate system, deg
Θ_{ir}	Initial angle between vertical line and rope, deg
$\mathbf{\Phi}$	Coriolis joint torque vector
ϕ_i	Angle between line segment C_iI' and $\boldsymbol{y}$-axis, deg

Latin Symbols

$\mathbf{a}_{ij}$	Vector between CoM of i-th link and j-th link
$\mathbf{a}_{ik}^{jl}$	Vector between CoM of i-th link of j-th module and CoM of k-th link of l-th module
$\mathbf{ac}_{r,t}$, $\mathbf{ac}_{l,t}$	Linear accelerations of torso at hip joint origins connected to right and left leg, respectively
$\mathbf{ac}_t$	Linear acceleration at CoM of torso, m/s^2
$(ac_x;ac_y)$	Linear acceleration components of point with coordinates $(x;y)$, m/s^2
$(ac_{xi};ac_{yi})$	Linear acceleration components of point with coordinates$(x_i;y_i)$, m/s^2
$(ac_{xci};ac_{yci})$	Linear acceleration components of i-th point of contact C_i represented in $\boldsymbol{x}_i\boldsymbol{y}_i$ coordinate frame, m/s^2
A	Arbitrary point in equilibrium region
A_i	Arbitrary point in zone Z_i on equilibrium line
$\underline{\mathbf{A}}_{ik}^{jl}$	Dual twist propagation matrix between i-th link of j-th module and k-th link of l-th module
$\underline{\mathbf{A}}_{ik}^{j}$	Dual twist propagation matrix between i-th and k-th link of j-th module.
$\underline{\mathbf{A}}^{jl}$	Dual twist propagation matrix between j-th and l-th module
Adj	Adjoint transformation matrix
$\mathbf{b}$	Single-column vector with real numbers as its elements
B_x, B_y	Points of intersection between inflection circle and $\boldsymbol{x}_1$, $\boldsymbol{y}_1$ axis respectively
c_{sij}	Element of swing leg MCI's i-th row and j-th column, N-m-s/rad
c_{ij}	Element of MCI's i-th row and j-th column, N-m-s/rad
c_{iz}^{j}	Element of MCI (of j-th module)'s i-th row and z-th column, N-m-s/rad
$\overline{c}_{iz}^{j}$	Element of RMCI (of j-th module)'s i-th row and z-th column, N-s/rad
$\mathbf{c}_i$	Vector from CoM of biped to CoM of i-th link

$\mathbf{cv}^j$	VCI of j-th module
$\overline{\mathbf{cv}}^j$	RVCI of j-th module
$\mathbf{C}^j$	MCI of j-th module
$\overline{\mathbf{C}}^j$	RMCI of j-th module
C	Point of contact (single contact)
C_i	Point of i-th contact (multiple contacts)
C_m, C_f	Centre of curvature of m-curve and f-curve respectively
C_{im}, C_{if}	Centre of curvature of i-th m-curve and f-curve respectively
C_Δ	Coefficient of t in $\Delta(t)$
CG	CG point
d_i	Signed distance from i-th joint to CoM of i-th link, m
D_ϵ	Dual operator, m
D_Δ	Coefficient of t^2 in $\Delta(t)$
$\mathbf{d}_i^j$	Vector from origin of i-th link of j-th module to CoM of the same link
$\mathbf{d}_{lj,li}$	Vector from origin of i-th link of left leg to the CoM of j-th link of same leg
$\mathbf{d}_{rj,ri}$	Vector from origin of i-th link of right leg to CoM of j-th link of same leg
$\underline{\mathbf{D}}$	Extraction operator matrix
$Du(\underline{\mathbf{z}})$	Dual part of dual vector $\underline{\mathbf{z}}$
$\mathbf{e}_i^j$	Axis of rotation for i-th link of j-th module
$\mathbf{e}_{ri}, \mathbf{e}_{li}$	Axis of rotation of i-th joint of right and left leg respectively
$\underline{\mathbf{E}}$	Dual coefficient matrix
e	Equilibrium line
$\mathbf{f}_a$	Externally applied force
$\mathbf{f}_{al}$	Force on base of biped due to motions and applied forces on other links
$\mathbf{f}_c$	Resultant contact force
$\mathbf{f}_{ci}$	Resultant contact force at i-th point of contact
$\mathbf{f}_{ci,aj}$	Ground contact force on i-th contact point of j-th link
$\mathbf{f}_f$	Friction force
$\mathbf{f}_{fi}$	Friction force at i-th point of contact
$\mathbf{f}_g$	Body force due to gravity
$\mathbf{f}_i^j$	Applied force at CoM of the i-th link of j-th module
$\mathbf{f}_n$	Normal contact force
$\mathbf{f}_{ni}$	Normal contact force at i-th point of contact

© 2024, *Indian Institute of Technology Delhi*

$\mathbf{f}_R$	Resultant of $\mathbf{f}_a$ and $\mathbf{f}_g$
$\overline{\mathbf{f}_r}^j$	JARF vector of j-th module
$\overline{\mathbf{f}_p}^j$	Contribution by parent module on JARF of j-th module
f_a	Line of action of $\mathbf{f}_a$
f_c	Line of action of $\mathbf{f}_c$
f_{ci}	Line of action of $\mathbf{f}_{ci}$
f_f	Line of action of $\mathbf{f}_f$
f_{fi}	Line of action of $\mathbf{f}_{fi}$
f_g	Line of action of $\mathbf{f}_g$
f_n	Line of action of $\mathbf{f}_n$
f_{ni}	Line of action of $\mathbf{f}_{ni}$
f_R	Line of action of $\mathbf{f}_R$
$\widehat{\mathbf{f}_\mathbf{a}}$	Unit vector along line f_a
$\widehat{\mathbf{f}_\mathbf{c}}$	Unit vector along line f_c
$\widehat{\mathbf{f}_\mathbf{ci}}$	Unit vector along line f_{ci}
$\widehat{\mathbf{f}_\mathbf{f}}$	Unit vector along line f_f
$\widehat{\mathbf{f}_\mathbf{fi}}$	Unit vector along line f_{fi}
$\hat{\mathbf{f}}_g$	Unit vector along line f_g
$\widehat{\mathbf{f}_\mathbf{n}}$	Unit vector along line f_n
$\widehat{\mathbf{f}_\mathbf{ni}}$	Unit vector along line f_{ni}
$\overline{f_{ri}}$	JARF vector's i-th element, N
$\overline{f_{ri}}^j$	JARF vector (of j-th module)'s i-th element, N
f_{n_i}	magnitude of normal contact force at i-th point of contact, N
$\mathbf{g}$	Acceleration vector due to gravity, m/s^2
g	Magnitude of acceleration due to gravity, 9.81 m/s^2
$\underline{\tilde{\mathbf{G}}}$	Dual matrix, used to calculate GIM and RIM
$\mathbf{h}$	Linear momentum
$\underline{\mathbf{h}}$	Dual momentum
h_{ij}	Element of GIM's i-th row and j-th column, kg-m^2
h_{sij}	Element of swing leg GIM's i-th row and j-th column, kg-m^2
h_{iz}^j	Element of GIM (of j-th module)'s i-th row and z-th column, kg-m^2
$\overline{h}_{iz}^j$	Element of RIM (of j-th module)'s i-th row and z-th column, kg-m
$\mathbf{H}$	GIM of whole body
$\mathbf{H}^j$	GIM of j-th module
$\overline{\mathbf{H}}^j$	RIM of j-th module

© 2024, *Indian Institute of Technology Delhi*

H	Hip joint location
$\#i^j$	Link number i of j-th module
i_i	Component of difference in i-th joint velocity after ground impact
i_b	Applied impulse on biped by ball
$\mathbf{I}$	Mass moment of inertia tensor of a rigid body about an axis passing through its CoM
$\mathbf{I}_i$	Mass moment of inertia of i-th link about an axis passing through its CoM
$\mathbf{I}_i^j$	Mass moment of inertia tensor of i-th link of j-th module about an axis passing through the its CoM
I	Velocity pole point
I_i	ICC of A_i
I_{xx}, I_{yy}, I_{zz}	Mass moment of inertia calculated in $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$-axis, respectively, kg-m^2
$I_{xx,b}$, $I_{yy,b}$, $I_{zz,b}$	Mass moment of inertia calculated in body-fixed frame, kg-m^2
$\mathbf{J}_a$	Jacobian matrix corresponding to applied external push force
$\mathbf{J}_c$	Jacobian matrix corresponding to contact force
$\mathbf{J}_h$	Linear momentum jacobian matrix of whole body
$\mathbf{J}_{rh}, \mathbf{J}_{lh}$	Linear momentum Jacobian matrix of right leg and left leg respectively
$\mathbf{J}_{rhi}, \mathbf{J}_{lhi}$	$\mathbf{J}_{rh}$ and $\mathbf{J}_{lh}$'s i-th column, respectively
J_i	Point of intersection of inflection circle with f_{ni}
J_g	Point of intersection of inflection circle with f_g
k_p	Proportional gain for PD control and CTC, N-m/rad (PD) and /s^2 (CTC)
k_d	Derivative gain for PD control and CTC, N-m-s/rad (PD) and /s (CTC)
$\mathbf{K}_p, \mathbf{K}_d$	Diagonal matrix with diagonal elements as k_p and k_d, respectively, for PD control and CTC
K	Knee joint location
$\mathbf{l}$	Angular momentum vector
l_i	Length of i-th link, m
ln	DoF of left leg
m	Mass, kg
m_b	Mass of ball, kg
m_i	Mass of i-th link, kg

© 2024, *Indian Institute of Technology Delhi*

m_i^j	Mass of i-th link of j-th module, kg
m_r, m_l, m_{tot}	Total mass of right leg, left leg, and whole biped, respectively, kg
m_{ri}, m_{li}	Mass of i-th link of right and left leg, respectively, kg
$\underline{\mathbf{M}}$	Dual inertia matrix
$\underline{\mathbf{M}}_i^j$	Dual inertia matrix of i-th link of j-th module
$\underline{\mathbf{M}}^j$	Dual inertia matrix of j-th module
$\tilde{\underline{\mathbf{M}}}^j$	DCBI of j-th module
$\#M_j$	Module number j
$\mathfrak{M}(t)$	Motion at time t
$\mathbf{n}_{al}$	Moment applied on base due to motions and applied forces and moments on other links
$\mathbf{n}_i^j$	Moment applied on CoM of i-th link of j-th module
nc	Number of contact points between foot and ground
nj	Number of joints
nl	Number of links
$\underline{\mathbf{N}}$	Dual DeNOC matrix of whole body
$\underline{\mathbf{N}}_d$	Dual block diagonal DeNOC matrix of whole body
$\underline{\mathbf{N}}_l$	Dual lower triangular DeNOC matrix of whole body
$\underline{\mathbf{N}}^j$	Dual DeNOC matrix of j-th module
$\underline{\mathbf{N}}_d^j$	Dual block diagonal matrix of j-th module
$\underline{\mathbf{N}}_l^j$	Dual lower block-triangular matrix of j-th module
$\mathbf{o}$	Orthogonal vector from C_1 upon line f_g
$\mathbf{o}_a$	Orthogonal vector from C_1 upon line f_a
$\mathbf{O}$	Zero matrix of compatible size
$\mathbf{o}_i$	Orthogonal vectors from C_1 upon line f_{ni}
O	Origin of moving coordinate system
Of	Objective function of the proposed optimization
$\underline{\mathbf{p}}_i^j$	Dual axis representation for i-th link of j-th module
p_{x_i}, p_{y_i}	$\boldsymbol{x}$ and $\boldsymbol{y}$-coordinates of position vectors of i-th contact point
P_x,P_y	Projections of B_x and B_y on $\boldsymbol{y}$-axis respectively
$\mathbf{q}_{ij}$	Position vector of C_j with respect to C_i
$\tilde{\mathbf{q}}_{ij}$	Skew-symmetric matrix form of $\mathbf{q}_{ij}$
$\dot{\mathbf{q}}^0$	Dual twist vector of torso
$\mathbf{Q}_{ij}$	Rotation matrix between coordinate frames $\boldsymbol{x}_j\boldsymbol{y}_j\boldsymbol{z}_j$ and $\boldsymbol{x}_i\boldsymbol{y}_i\boldsymbol{z}_i$
$\mathbf{r}_{aj}$	Vector from CoM of biped to j-th ankle joint

© 2024, *Indian Institute of Technology Delhi*

$\mathbf{r}_{ci,aj}$	Vector from j-th ankle joint origin to i-th contact point of foot of same leg
rn	DoF of right leg
$\underline{\mathbf{R}}$	Duality inducing matrix
$\mathbf{R}_{s1}, \mathbf{R}_{s2}$	Symmetric matrices of size 3×3
$\mathbf{R}_{ss}$	Skew symmetric matrix of size 3×3
$\mathbf{R}$	Representation of $\underline{\mathbf{R}}$ as a 6×6 matrix
$Re(\underline{\mathbf{z}})$	Real part of dual vector $\underline{\mathbf{z}}$
s	Signed distance of CG from velocity pole point, m
s_0	Signed distance from C_1 to line f_a, m
s_1, s_2	Signed distances from C_1 and C_2 to velocity pole O respectively, m
s_c	Swing leg clearance during walking, m
s_l	Step-length of walking, m
s_{lim}	Limiting value of s, m
s_r	Length of rope, m
s_x	Distance between two contact points along $\boldsymbol{x}$-axis, m
s_y	Distance between two contact points along $\boldsymbol{y}$-axis, m
sym	Symmetric block elements of matrix
$\mathbf{t}_i$	Twist vector at $\boldsymbol{x}_i\boldsymbol{y}_i\boldsymbol{z}_i$ coordinate system
$\underline{\mathbf{t}}$	Dual twist vector for whole body
$\underline{\mathbf{t}}^j$	Dual twist vector for j-th module
$\underline{\mathbf{t}}_i^j$	Dual twist vector for i-th link of j-th module
t	Time, s
T	Time period of walking, s
$\hat{\mathbf{u}}_{t1}$, $\hat{\mathbf{u}}_{t2}$, $\hat{\mathbf{u}}_n$	Unit vectors along two mutually perpendicular directions on ground and along normal directions to ground, respectively
$\mathbf{v}$	Linear velocity vector
$\mathbf{v}_{b0}$	Linear velocity of ball while hitting biped
$\mathbf{v}_{cm}$	Linear velocity of CoM of whole biped
$\mathbf{v}_i$	CoM Velocity of i-th link
$\mathbf{v}_i^j$	Linear velocity of CoM of i-th link of j-th module
$\mathbf{v}_{r,t}$, $\mathbf{v}_{l,t}$	Linear velocity of hip joint origins of right and left leg, respectively
$\mathbf{v}_t$	Linear velocity of CoM of torso
$\boldsymbol{v_d}$	Desired joint velocity vector for push recovery

© 2024, *Indian Institute of Technology Delhi*

$\mathbf{V}$	Matrix whose columns are made up of those eigenvectors of matrix $\mathbf{J}_h^T\mathbf{J}_h$ that correspond to zero eigenvalues
$(v_x;v_y)$	Linear velocity components of point with coordinates $(x;y)$, m/s
$(v_{xi};v_{yi})$	Linear velocity components of point with coordinates $(x_i;y_i)$, m/s
$(v_{xci};v_{yci})$	Linear velocity components of C_i represented in $\boldsymbol{x}_i\boldsymbol{y}_i$ coordinate frame, m/s
w	Weight of biped, N
$\underline{\mathbf{w}}$	Dual force vector on whole body
$\underline{\mathbf{w}}^j$	Dual force vector on j-th module
$\underline{\mathbf{w}}_i^j$	Dual force vector on CoM of i-th link of j-th module
$\underline{\mathbf{w}}^C$	Dual constraint force vector
$\underline{\mathbf{w}}^E$	Dual external force vector due to input force/torque
$\underline{\mathbf{w}}^G$	Dual external force vector due to gravity
$\underline{\mathbf{w}}^{\mathbf{C}}$, $\underline{\tilde{\mathbf{w}}}_i'$	Component of input joint torque and JARF
$\underline{\mathbf{W}}$	Component of MCI and RMCI of whole biped
$\underline{\mathbf{W}}^j$	Component of MCI and RMCI of j-th module
$\underline{\mathbf{W}}_i^j$	Component of MCI and RMCI of i-th link of j-th module
$\underline{\mathbf{z}}, \underline{\mathbf{z}}_1, \underline{\mathbf{z}}_2$	Dual vectors
$\mathbf{z}_d$, $\mathbf{z}_{d1}$, $\mathbf{z}_{d2}$	Dual part of $\underline{\mathbf{z}}, \underline{\mathbf{z}}_1$ and $\underline{\mathbf{z}}_2$ respectively
$\mathbf{z}_r$, $\mathbf{z}_{r1}$, $\mathbf{z}_{r2}$	Real parts of $\underline{\mathbf{z}}, \underline{\mathbf{z}}_1$ and $\underline{\mathbf{z}}_2$ respectively
Z_i	Zone i on equilibrium line
ZMP_x, ZMP_y	$\boldsymbol{x}$ and $\boldsymbol{y}$-coordinates of ZMP, m
$\mathbf{0}$	Zero vector of compatible size
$\mathbf{1}$	Identity matrix of compatible size

© 2024, *Indian Institute of Technology Delhi*