

**HARMONIC ANALYSIS ON ORLICZ SPACES FOR CERTAIN
HYPERGROUPS AND ON DISCRETE HYPERGROUPS
ARISING FROM SEMIGROUPS WITH EMPHASIS ON
RAMSEY THEORY**

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INDIAN INSTITUTE OF TECHNOLOGY DELHI
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ARISING FROM SEMIGROUPS WITH EMPHASIS ON
RAMSEY THEORY**

by

VISHVESH KUMAR

Department of Mathematics

Submitted

in fulfillment of the requirements of the degree of Doctor of Philosophy

to the



**Indian Institute of Technology Delhi
April 2019**

Dedicated to
My Brother “Yogesh Kumar Mishra”

Certificate

This is to certify that the thesis entitled **Harmonic analysis on Orlicz spaces for certain hypergroups and on discrete hypergroups arising from semigroups with emphasis on Ramsey theory** submitted by **Mr. Vishvesh Kumar** to the **Indian Institute of Technology Delhi**, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bona fide research work carried out by him under our guidance and supervision. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

April 2019

New Delhi

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New Delhi
April, 2019

Vishvesh Kumar

Abstract

This thesis is dedicated to the study of a new method for constructing a family of hypergroups or semi convolution spaces, Ramsey theory for hypergroups and the harmonic analysis on Orlicz spaces on hypergroups.

We introduce a method to construct a family of hypergroups or semi convolution spaces from general commutative semigroups. For this purpose, we view the well-known example of the dual of a countable compact hypergroup, motivated by the orbit space of p -adic integers by Dunkl and Ramirez (1975), as hypergroup deformation of the max semigroup structure on the linearly ordered set \mathbb{Z}_+ of the non-negative integers along the diagonal. This motivates us to study hypergroups or semi convolution spaces arising from “max” semigroups or general commutative semigroups via hypergroup deformation on idempotents.

We present a systematic study of Ramsey theory for discrete hypergroups with emphasis on polynomial hypergroups, discrete orbit hypergroups and hypergroup deformations of semigroups. In this context, new notions of Ramsey principle for hypergroups and α -Ramsey hypergroup, $0 \leq \alpha < 1$, are defined and studied. We also give several examples to distinguish them from each other.

We also begin a study of harmonic analysis on Orlicz spaces on hypergroups. For a locally compact hypergroup K and a Young function Φ , we study the Orlicz space $L^\Phi(K)$ and provide a sufficient condition for $L^\Phi(K)$ to be an algebra under

convolution of functions. We show the existence of an approximate identity in the Orlicz algebra on a hypergroup and as an application we give a characterization of its left ideals. We show that there is no bounded approximate identity in any Orlicz algebra on a non-discrete hypergroup. We present certain results related to characterization of multipliers of the Morse-Transue space $M^\Phi(K)$ and $L^\Phi(\mathcal{S}, \pi_K)$, where \mathcal{S} is the support of the Plancherel measure π_K associated to a commutative hypergroup K . Finally, we introduce Rao-Reiter condition (P_Φ) and its variants to study the amenability of hypergroups.

At last, we prove the classical Hausdorff-Young inequality for the Lebesgue spaces on a compact hypergroup using interpolation of sublinear operators. We use this result to prove the Hausdorff-Young inequality for Orlicz spaces on a compact hypergroup.

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सार

यह शोध प्रबंध हाइपरग्रुप्स या सेमी कन्वोलुशन स्पेसेस के एक फैमिली के निर्माण के लिए एक नई विधि के अध्ययन, हाइपरग्रुप्स के लिए रैमसे सिद्धांत और हाइपरग्रुप्स पर ओर्लिकज स्पेसेस पर हार्मोनिक विश्लेषण के लिए समर्पित है।

हम सामान्य कम्यूटेटिव सेमीग्रुप्स से हाइपरग्रुप्स या सेमी कन्वोलुशन स्पेसेस के फैमिली का निर्माण करने के लिए एक विधि शुरू करते हैं। इस प्रयोजन के लिए, हम पी-एडिक पूर्णांकों की ऑर्बिट स्पेस से प्रेरित, डंकल और रामिरेज़ (१९७५) के द्वारा दिए गए एक गणनीय कॉम्पैक्ट हाइपरग्रुप के डुअल के प्रसिद्ध उदाहरण को, रैखिक रूप से क्रमिक नॉन-नेगेटिव पूर्णांक Z_+ के “मैक्स” सेमीग्रुप संरचना को विकर्ण के साथ हाइपरग्रुप डिफ़ॉर्मेशन के रूप में देखते हैं। यह हमें ऑयडेम्पोटेंट पर हाइपरग्रुप डिफ़ॉर्मेशन के माध्यम से “मैक्स” सेमीग्रुप या सामान्य कम्यूटेटिव सेमीग्रुप्स से उत्पन्न हाइपरग्रुप्स या सेमी कन्वोलुशन स्पेसेस का अध्ययन करने के लिए प्रेरित करता है।

हम पोलीनोमिअल हाइपरग्रुप्स, डिस्क्रीट ऑर्बिट हाइपरग्रुप्स और सेमीग्रुप के हाइपरग्रुप डिफ़ॉर्मेशन पर जोर देने के साथ डिस्क्रीट हाइपरग्रुप्स के लिए रैमसे सिद्धांत का एक व्यवस्थित अध्ययन प्रस्तुत करते हैं। इस संदर्भ में, हाइपरग्रुप्स के लिए रैमसे सिद्धांत की नई धारणाएं और α -रैमसे हाइपरग्रुप, $0 \leq \alpha < 1$, परिभाषित और अध्ययन की गई हैं। हम उन्हें एक दूसरे से अलग करने के लिए कई उदाहरण भी देते हैं।

हम हाइपरग्रुप्स के लिए ओर्लिकज स्पेसेस पर हार्मोनिक विश्लेषण का अध्ययन भी शुरू करते हैं। हम लोकली कॉम्पैक्ट हाइपरग्रुप K और यंग फंक्शन Φ के लिए ओर्लिकज स्पेस $L^\Phi(K)$ का अध्ययन करते हैं और $L^\Phi(K)$ के लिए फंक्शनों के कन्वोलुशन के तहत एक अलजेब्रा होने के लिए पर्याप्त शर्त प्रदान करते हैं। हम एक हाइपरग्रुप पर ओर्लिकज अलजेब्रा में एक अप्रोक्सिमेट आइडेंटिटी के अस्तित्व को दिखाते हैं और इसका उपयोग करके हम इसके लेफ्ट आइडियल्स का वर्णन करते हैं। हम दिखाते हैं कि नॉन-डिस्क्रीट हाइपरग्रुप पर किसी भी ओर्लिकज अलजेब्रा में कोई अप्रोक्सिमेट आइडेंटिटी नहीं है। हम मोर्स-ट्रान्स स्पेस $M^\Phi(K)$ और $L^\Phi(S, \pi_K)$ के मल्टिप्लायर्स के वर्णन से संबंधित कुछ परिणाम प्रस्तुत करते हैं, जहाँ S एक कम्प्यूटेटिव हाइपरग्रुप K से सम्बंधित प्लांचेरेल मेजर π_K का सपोर्ट है। आखिर में, हम हाइपरग्रुप्स की एमेनबिलिटी का अध्ययन करने के लिए राव-राइटर कंडीशन (P_Φ) और इसके वेरिएंट का परिचय देते हैं।

अंत में, हम सबलीनियर ऑपरेटर्स के इन्टरपोलेशन का उपयोग करके एक कॉम्पैक्ट हाइपरग्रुप पर लेबेग स्पेसेस के लिए क्लासिकल हॉसडोर्फ-यंग इनक्वॉलिटी साबित करते हैं। हम इस परिणाम का उपयोग एक कॉम्पैक्ट हाइपरग्रुप पर ओर्लिकज स्पेसेस के लिए हॉसडोर्फ-यंग इनक्वॉलिटी को साबित करने के लिए करते हैं।

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List of Symbols

Symbol Meaning

\forall	for all
$=$	equal to
\neq	not equal to
\in	belongs to
\notin	does not belong
\subset	subset or equal
\cup, \cap	union, intersection
$ x $	absolute value of x
\cong	isomorphic to
\emptyset	empty set

\mathbb{N}	the set of natural numbers
\mathbb{Z}	the set of integers
\mathbb{Z}_+	the set of non-negative integers
\mathbb{Q}	the set of rational numbers
\mathbb{R}	the real line
$\mathcal{P}(X)$	the power set of a set X
X	the locally compact Hausdorff space
$C(X)$	the space of all complex-valued continuous functions on X
$C^b(X)$	the space of all bounded continuous functions X
$C_c(X)$	the space of compactly support continuous functions X
$C_0(X)$	the space of continuous functions X vanishing at infinity
$M(X)$	the space of all complex-valued bounded Borel regular measures on X
$M_p(X)$	the subset of $M(X)$ consisting of all probability measures
δ_x	the unit point mass measure at x or the Dirac-delta measure at x
$\text{supp}(f)$	the support of a complex-valued function f on X
$\text{supp}(\mu)$	the support of a measure μ in $M(X)$
$\mu(\{j\})$	the measure of singleton set $\{j\} \subset X$ with respect to μ , i.e., $\mu(\{j\})$.
K	a hypergroup
λ	a Haar measure on K
\widehat{K}	the dual object of a hypergroup
π_K	the Plancherel measure of on the dual \widehat{K} of a abelian hypergroup K
\mathcal{S}	the support of π_K
$\mu * \nu$	the convolution product of two measures μ and ν in $M(K)$
\widehat{f}	the Fourier transform of f
\square	end of a proof