BILLIARDS ON HYPERBOLIC TABLES

PRADEEP SINGH

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY DELHI

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by

Pradeep Singh
Department of Mathematics

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Dedicated to
my family

Certificate

I certify that the thesis entitled ***Billiards on Hyperbolic Tables*** presented by **Mr. Pradeep Singh** (2017MAZ8308) is worthy of consideration for the award of the degree of **Doctor of Philosophy** and is a record of the original bonafide research work carried out by him under my guidance and supervision and the results contained in it have not been submitted in part or full to any other university or institute for the award of any degree/diploma.

Prof. Anima Nagar
Department of Mathematics
Indian Institute of Technology Delhi

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Abstract

J. HADAMARD studied the geometric properties of geodesic flows on the surfaces of negative curvature, thus initiating "Symbolic Dynamics". We build upon the same geometric approach to study the geodesic trajectories of billiards in "ideal", "rational" and "semi-ideal rational" polygons on the hyperbolic plane. We particularly study the resulting billiard dynamics which comes out to be just the 'Subshifts of Finite Type'(SFT) or their dense subsets.

Using the results obtained in the hyperbolic plane, we further probe the hyperbolic 3-space for the same. We produce the coding rules of billiards for a special class of ideal polyhedrons in the 3-dimensional hyperbolic space establishing conjugacy between the space of pointed billiard trajectories and the corresponding shift space of codes. Interestingly, the closure of the related shift space again comes out to be an SFT.

Further, we take up a study involving decisive Bratteli-Vershik models for polygonal billiards on the hyperbolic plane, which have been previously studied for compact, invertible zero-dimensional systems. So, we consider these models beyond zero-dimension. We describe the associated Bratteli models and show that these billiard dynamics can be described by a Vershik map.

सार

जे हैडामर्ड ने नकारात्मक वक्रता की सतहों पर जियोडेसिक प्रवाह के ज्यामितीय गुणों का अध्ययन किया, इस प्रकार "सिम्बोलिक डायनामिक्स" की शुरुआत हुई। हम, उसी ज्यामितीय पद्धति का अनुकरण करते हुए, हाइपरबोलिक विमान पर "आदर्श", "तर्कसंगत" और "अर्ध-आदर्श तर्कसंगत" बहुभुजों में बिलियर्ड्स के भू-आदर्श प्रक्षेपवक्रों का अध्ययन करते हैं । हम विशेष रूप से परिणामी बिलियर्ड गतिशीलता का अध्ययन करते हैं जो सिर्फ 'परिमित प्रकार के उप-शिफ्ट' (एसएफटी) या उनके सघन उपसम्मुचय के रूप में सामने आता है।

हाइपरबोलिक विमान में प्राप्त निष्कर्षों का प्रयोग करके, हम आगे इसके लिए हाइपर-बोलिक 3-स्पेस की जांच करते हैं। हम त्रि-आयामी हाइपरबोलिक स्पेस में आदर्श पॉलीहेड्रॉन के एक विशेष वर्ग के लिए बिलियर्ड्स के कोडिंग नियमों का निर्माण करते हैं, जो नुकीले बिलियर्ड प्रक्षेपवक्रों के स्थान और कोड से संबंधित शिफ्ट स्पेस के बीच संयुग्म (कॉज्युगेसी) स्थापित करता हैं। दिलचस्प बात यह है कि संबंधित शिफ्ट स्पेस का बंद होना पुनः एक एसएफटी के रूप में सामने आता है।

इसके अलावा, हम हाइपरबोलिक विमान पर पॉलीगॉनल बिलियर्ड्स के लिए निर्णायक ब्रैटेली-वरसिक मॉडल से जुड़े एक अध्ययन को लेते हैं, जिसका अध्ययन पहले संक्षिप्त एवं प्रतिप्य शून्य-आयामी प्रणालियों के लिए किया गया है। इसलिए, हम इन मॉडलों को शून्य-आयाम से परे मानते हैं। हम संबंधित ब्राटेली मॉडल का वर्णन करते हुए ये दर्शाते हैं कि इन बिलियर्ड गतिशीलताओं को वर्शिक मानचित्र द्वारा निरूपित किया जा सकता है।

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