

ASSOCIATED LIE ALGEBRAS OF GROUP ALGEBRAS

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ASSOCIATED LIE ALGEBRAS OF GROUP ALGEBRAS

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Dedicated to
My Family

Certificate

This is to certify that the thesis entitled “**Associated Lie Algebras of Group Algebras**” submitted by **Ms Reetu Siwach** to the Indian Institute of Technology Delhi, for the award of the degree of **Doctor of Philosophy**, is a record of the original bona fide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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Abstract

This thesis is a study of associated Lie algebras of group algebras. We are particularly interested in Lie nilpotent and Lie solvable group algebras. Passi, Passman and Sehgal obtained necessary and sufficient conditions for a group algebra to be Lie nilpotent and Lie solvable. Chandra and Sahai classified Lie nilpotent group algebras with nilpotency index at most 8. Motivated by this, we have given a classification of Lie nilpotent group algebras with nilpotency index 9, 10, 11, 12 and 13. This study is distributed over two chapters namely Chapter 2 and Chapter 3. In Chapter 2, we have classified Lie nilpotent group algebras with upper Lie nilpotency index 9, 10 and 11. In Chapter 3, Lie nilpotent group algebras with upper Lie nilpotency index 12 and 13 are investigated.

Let KG be the group algebra of a group G over a field K . For a Lie nilpotent group algebra KG , upper bound for the Lie nilpotency index $t^L(KG)$ was given by Sharma and Bist. In the last 10 years many authors like Bovdi, Juhasz, Spinelli and Srivastava have worked on the maximal, almost maximal and next possible Lie nilpotency indices of group algebras. Thus the characterization of group algebras KG with $t^L(KG) = |G'| + 1$, $|G'| - p + 2$, $|G'| - 2p + 3$, $|G'| - 3p + 4$ and $|G'| - 4p + 5$ is well known. Continuing this study, we have obtained classification for Lie nilpotent group algebras KG with $t^L(KG) = |G'| - 5p + 6$, $|G'| - 6p + 7$, $|G'| - 7p + 8$, $|G'| - 8p + 9$, $|G'| - 9p + 10$, $|G'| - 10p + 11$, $|G'| - 11p + 12$,

$|G'| - 12p + 13$ and $|G'| - 13p + 14$. This classification extends in two chapters namely Chapter 4 and Chapter 5. In Chapter 4, we have classified Lie nilpotent group algebras with $t^L(KG) = |G'| - 5p + 6, |G'| - 6p + 7, |G'| - 7p + 8$. In Chapter 5, we have given a classification of Lie nilpotent group algebras with $t^L(KG) = |G'| - 8p + 9, |G'| - 9p + 10, |G'| - 10p + 11, |G'| - 11p + 12, |G'| - 12p + 13$ and $|G'| - 13p + 14$.

In this thesis, we have also focused on the derived length of Lie solvable group algebras. After the classification of Lie solvable group algebras by Passi, Passman and Sehgal; Levin and Rosenberger characterized Lie metabelian group algebras. Further Lie centrally metabelian group algebras have been studied by Külshammer, Sharma, Rossmanith, Sahai and Srivastava. After this Sahai and Chandra studied Lie solvable group algebras of derived length 3 for odd characteristic. Motivated by this we have classified strongly Lie solvable group algebras of derived length 4 whenever characteristic of the field K is $p \geq 7$.

सार

यह शोधग्रंथ ग्रुप अल्जेब्रा के असोसिएटेड ली अल्जेब्रा का अध्ययन है। हमने ली निल्पोटेंट ग्रुप अल्जेब्रा और ली सॉल्वेबल ग्रुप अल्जेब्रा का वर्णन किया है। शोधकर्ताओं श्रीमान पासी, पासमैन और सहगल ने ग्रुप अल्जेब्रा के ली निल्पोटेंट और ली सॉल्वेबल होने के लिए आवश्यक और पर्याप्त अनुबंध प्राप्त किये थे। चंद्रा और सहाय ने ली निल्पोटेंट ग्रुप अल्जेब्रा, जिनकी निल्पोटेंसी इंडेक्स अधिकतम 8 हैं, को वर्गीकृत किया है। इससे प्रेरित होकर हमने निल्पोटेंसी इंडेक्स 9, 10, 11, 12 और 13 वाले ली निल्पोटेंट ग्रुप अल्जेब्रा को वर्गीकृत किया है। यह वर्गीकरण इस शोधग्रंथ के अध्याय संख्या 2 व 3 में है। द्वितीय अध्याय में हमने निल्पोटेंसी इंडेक्स 9, 10, 11 वाले ली निल्पोटेंट ग्रुप अल्जेब्रा को वर्गीकृत किया है, और तृतीय अध्याय में हमने निल्पोटेंसी इंडेक्स 12 और 13 वाले ली निल्पोटेंट ग्रुप अल्जेब्रा को वर्गीकृत किया है।

मान लेते हैं कि KG , ग्रुप G के फ़िल्ड K पर, ग्रुप अल्जेब्रा को परिभाषित करता है। ली निल्पोटेंट ग्रुप अल्जेब्रा KG के लिए, ली निल्पोटेंसी इंडेक्स की उत्तम सीमा $t^L(KG)$ का शर्मा व बिष्ट ने अध्ययन किया है। पिछले दस वर्षों में अनेक शोधकर्ताओं बोवडी, जुहाशज़, स्पाइनेली, श्रीवास्तवा इत्यादि ने ली निल्पोटेंसी इंडेक्स की सीमा का काफ़ी अध्ययन किया है। इस क्रम में हमने $t^L(KG) = |G'| - 5p + 6$, $t^L(KG) = |G'| - 6p + 7$, $t^L(KG) = |G'| - 7p + 8$, $t^L(KG) = |G'| - 8p + 9$, $t^L(KG) = |G'| - 9p + 10$, $t^L(KG) = |G'| - 10p + 11$, $t^L(KG) = |G'| - 11p + 12$, $t^L(KG) = |G'| - 12p + 13$, $t^L(KG) = |G'| - 13p + 14$ वाले ली निल्पोटेंट ग्रुप अल्जेब्रा को वर्गीकृत किया है। इस वर्गीकरण को पुनः हमने दो अध्यायों क्रमशः अध्याय 4 व अध्याय 5 में वर्गीकृत किया है।

अध्याय 4 में हमने निल्पोटेंसी इंडेक्स $t^L(KG) = |G'| - 5p + 6$, $t^L(KG) = |G'| - 6p + 7$, $t^L(KG) = |G'| - 7p + 8$ वाले ली निल्पोटेंट ग्रुप अल्जेब्रा को वर्गीकृत किया है। अध्याय 5 में हमने उन निल्पोटेंट ग्रुप अल्जेब्रा को वर्गीकृत किया है, जिनके लिए निल्पोटेंसी इंडेक्स $t^L(KG) = |G'| - 8p + 9$, $t^L(KG) = |G'| - 9p + 10$, $t^L(KG) = |G'| - 10p + 11$, $t^L(KG) = |G'| - 11p + 12$, $t^L(KG) = |G'| - 12p + 13$, $t^L(KG) = |G'| - 13p + 14$ हैं।

इस शोधग्रंथ में हमने ली सॉल्वेबल ग्रुप अल्जेब्रा की डिग्री लेंथ पर भी ध्यान केन्द्रित किया है। पासी, पासमैन व सहगल के द्वारा ली सॉल्वेबल ग्रुप अल्जेब्रा को वर्गीकृत करने के पश्चात, लेविन व रोजेनबर्गर ने मेटाबेलियन ग्रुप अल्जेब्रा को वर्गीकृत किया है। तत्पश्चात कुलशमर, शर्मा, रोजमानिथ, सहाय और श्रीवास्तवा ने ली सेंट्रली मेटाबेलियन ग्रुप अल्जेब्रा का अध्ययन किया । तदुपरांत सहाय और चंद्रा का उद्देश्य भी ली सॉल्वेबल ग्रुप अल्जेब्रा, जिनकी डिग्री लेंथ 3 है, का अध्ययन रहा है। इससे प्रेरित होकर हमने स्ट्रॉंग ली सॉल्वेबल ग्रुप अल्जेब्रा, जिनकी डिग्री लेंथ 4 है, को वर्गीकृत किया है जहाँ पर फील्ड की कैरेक्टरीस्टिक >5 है।

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List of Symbols

$\mathbb{Z}$	the set of integers
$\mathbb{R}$	the set of real numbers
p	a prime number
$\forall x$	for all x
$o(x)$	The order of the group element x
$\hat{x}$	$1 + x + x^2 + \dots + x^{n-1}$, where $o(x) = n$
(x, y)	$x^{-1}y^{-1}xy$
$[x, y]$	$xy - yx$
$\langle X \rangle$	The subgroup generated by the subset X of a group
(X, Y)	The subgroup generated by all the commutators (x, y) , where $x \in X, y \in Y$
$H \leq G$	H is a subgroup of a group G
C_n	Cyclic group of order n
D_{2n}	Dihedral group of order $2n$
$ G $	Order of the group G
$\exp(G)$	Exponent of the group G
$\text{cl}(G)$	Nilpotency class of the group G
$N \leq G$	N is a normal subgroup of the group G
$\zeta(G)$	Center of the group G
$ \zeta(G) $	Order of the center of the group G

G'	The derived subgroup of the group G
$\delta^n(G)$	The n -th term of the derived series of the group G
$\gamma_n(G)$	The n -th term of the lower central series of the group G
RG	Group ring of a group G over a ring R
ϵ	Augmentation map
$\Delta_R(G)$	The augmentation ideal of RG
$t(G)$	Nilpotency index of $\Delta_R(G)$
$\text{Char}R$	Characteristic of the ring R
$L(R)$	Associated Lie ring of the ring R
$\delta^{(n)}(R)$	The n -th term of the strong Lie derived series of the ring R
$\delta^{[n]}(L(R))$	The n -th term of the Lie derived series of R
$\gamma_{(n)}(R)$	The n -th term of the strong Lie central series of R
$\gamma_{[n]}(R)$	The n -th term of the Lie central series of R
$D_{n,K}(G)$	The n -th dimension subgroup of G over K
$D_{[n],K}(G)$	The n -th restricted Lie dimension subgroup of G over K
$D_{(n),K}(G)$	The n -th Lie dimension subgroup of G over K
$S(n, m)$	The group number m of order n from the Small Groups Library GAP