# NUMERICAL SOLUTION OF SOME OPTION PRICING MODELS

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# Numerical Solution of Some Option Pricing Models

by

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*Submitted*

*in fulfillment of the requirements of the degree of Doctor of Philosophy*

*to the*

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**To**

**My Parents**

# Certificate

This is to certify that the thesis entitled **Numerical Solution of Some Option Pricing Models** submitted by **Ms. Manisha** to the Indian Institute of Technology Delhi, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by her under my supervision. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for award of any degree or diploma.

New Delhi
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# Abstract

Some of the European style option pricing models of the financial market are in the forms of: one dimensional degenerate parabolic partial differential equation, ultra-parabolic partial differential equation, partial integro-differential equation and two dimensional parabolic partial differential equation. When the real market complexities are taken into account, the closed form solutions of these models are not available. We develop some high-order convergent numerical methods for such problems.

A high-order difference approximation with identity expansion (HODIE) scheme along with two-step backward differentiation formula is applied to the generalized Black-Scholes option pricing model for four different European option styles which are degenerate parabolic partial differential equation problems. It is proved that the method has second order convergence in space as well as in time. This scheme is also applied to solve dimension reduced generalized Black-Scholes model for pricing Asian option. Further this scheme is extended to produce an accelerated convergence in space when applied to the log transformed generalized Black-Scholes partial differential equation problem.

The HODIE scheme along with Simpson's $\frac{1}{3}$rd formula and two-step backward differentiation formula, altogether as an implicit explicit scheme, are applied to solve generalized case of jump-diffusion model for pricing European put option numerically. The method has third order of accuracy in space and second order accuracy in time.

The Peaceman-Rachford Alternating Direction Implicit scheme for temporal semi-discretization and the HODIE scheme for spacial semi-discretization is proposed for numerically solving the two asset Black-Scholes model that is transformed into two dimensional diffusion equation, and the method is proved to be second order accurate in time and fourth order accurate in space.

A first order accurate Alternating Direction Implicit scheme, which is a two dimensional analogue to the backward Euler formula is developed and applied along with the HODIE scheme for solving the generalized model for Asian option which is governed by an ultra-parabolic two dimensional partial differential equation problem. The scheme is proved to be second order accurate in space.

Numerical experiments are in support of the theoretical results and demonstrate the effectiveness of the designed numerical methods.

# सारांश

वित्तीय बाज़ार में यूरोपीय शैली के ऑप्शन मूल्य निर्धारण तंत्रों में से कुछ निम्न रूप में हैं : एक आयामी अपकृष्ट परवलयिक आंशिक अवकल समीकरण, अल्ट्रा-परवलयिक आंशिक अवकल समीकरण, आंशिक समाकल-अवकल समीकरण तथा दो आयामी परवलयिक आंशिक अवकल समीकरण। जब वास्तविक बाज़ार की जटिलताओं को समाविष्ट किया जाता है, तब इन तंत्रों का सटीक हल उपलब्ध नहीं होता। हमनें इस प्रकार की समस्याओं के लिए कुछ उच्च ऑर्डर की अभिसृत संख्यात्मक विधियां विकसित की हैं।

चार विभिन्न यूरोपीय ऑप्शन शैलियों के व्यापक बनाए हुए ब्लैक-शोल्स ऑप्शन मूल्य निर्धारण तंत्र, जो कि अपकृष्ट परवलयिक आंशिक अवकल समीकरण समस्याएं हैं, पर एक हाई-ऑर्डर डिफरेंस अप्प्रोक्सिमेशन विद आइडेंटिटी एक्सपैंशन (HODIE) विधि सह द्वि-पदीय पश्चवर्ती अवकलन सूत्र लगाया गया है। यह प्रमाणित किया जाता है की उपरोक्त विधि का स्थान तथा समय में अभिसरण द्वितीय ऑर्डर का है। एशियाई ऑप्शन के मूल्य निर्धारण के आयाम घटाये हुए एवं व्यापक बनाए हुए ब्लैक-शोल्स तंत्र पर भी यह पद्धति लगाई गई है। तदोपरान्त इस विधि को स्थान में त्वरित अभिसरण बनाने के लिए विस्तारित किया जाता है जब इसे लघुगणक परिणत व्यापक बनाए हुए ब्लैक-शोल्स आंशिक अवकल समीकरण समस्या पर लागू किया जाता है।

सिम्पसन के 1/3 वें फॉर्मूला एवं द्वि-पदीय पश्चवर्ती अवकलन सूत्र के साथ HODIE विधि, सब मिला कर एक अंतर्निहित अपरोक्ष पद्धति के रूप में, यूरोपीय पुट ऑप्शन के मूल्य निर्धारण के लिए व्यापक बनाए हुए जम्प-डिफ्फुज़न तंत्र को संख्यात्मक रूप से हल करने के लिए उपयोग किया जाता है। इस विधि में स्थान में सटीकता का तीसरा ऑर्डर और समय में सटीकता का दूसरा ऑर्डर है।

सामयिक अर्ध-विच्छेदन के लिए पीसमैन-रचफोर्ड एकान्तरिक दिशा अंतर्निहित विधि और स्थानिक अर्द्ध-विच्छेदन के लिए HODIE विधि संख्यात्मक रूप से दो संपत्ति ब्लैक-शोल्स तंत्र को हल करने के लिए प्रस्तावित किया गया है जिसे दो आयामी विसरण समीकरण में रूपांतरित किया गया है, और विधि समय में दूसरे ऑर्डर की

सटीक तथा स्थान में चौथे ऑर्डर की सटीक साबित हुई है।

एक प्रथम ऑर्डर सटीक एकान्तरिक दिशा अंतर्निहित विधि, जो कि पश्चवर्ती यूलर सूत्र का एक द्वि-आयामी एनालॉग है, को विकसित किया गया है तथा HODIE विधि के साथ एशियाई ऑप्शन के व्यापक बनाए हुए तंत्र, जो की एक अल्ट्रा-परवलयिक द्वि-आयामी आंशिक अवकल समीकरण समस्या है, को हल करने के लिए लगाया गया है।

संख्यात्मक प्रयोग सैद्धांतिक परिणामों के समर्थन में हैं और रूपांकित संख्यात्मक विधियों की प्रभावशीलता प्रदर्शित करते हैं।

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# Notations

| | |
|---|---|
| $S$ | asset price |
| $S_0$ | initial asset price |
| $\tau, t$ | time variables |
| $C$ | call option price |
| $P$ | put option price |
| $\sigma$ | market volatility |
| $r$ | risk free interest rate |
| $D$ | dividend yield |
| $K$ | strike price |
| $T$ | maturity time of the contract |
| $W$ | Wiener process |
| $\varepsilon$ | small parameter |
| $a_2$ | diffusion coefficient |
| $a_1$ | convection coefficient |
| $a_0$ | reaction coefficient |
| $\alpha_i,\ i = -, c, +$ | HODIE coefficients corresponding to the stencil points |
| $\beta_i,\ i = -, c, +$ | HODIE coefficients corresponding to the auxiliary points |
| $h$ | grid length in space direction |
| $k$ | grid length in time direction |
| $M,\ N$ | mesh discretization parameters |

| | |
|---|---|
| $L$ | continuous differential operator |
| $\Omega_S,\ \Omega_x$ | space variable domain |
| $\Omega_t$ | variable domain |
| $\Omega$ | the cross product of space and time domains |
| $E_{\max}$ | maximum absolute error |
| $E_{\text{rms}}$ | root mean square error |
| $\|.\|$ | norm when domain is obvious, or of no particular significance |
| $v_i$ | $v(x_i)$ |
| $\mathcal{C},\ \mathcal{C}_s,\ s=0,1,\ldots$ | generic positive constants, independent of $M,\ N$ |