

# NUMERICAL SOLUTION OF SOME OPTION PRICING MODELS

MANISHA



DEPARTMENT OF MATHEMATICS  
INDIAN INSTITUTE OF TECHNOLOGY DELHI  
OCTOBER 2017

©Indian Institute of Technology Delhi (IITD), New Delhi, 2017

# Numerical Solution of Some Option Pricing Models

by

Manisha

Department of Mathematics

*Submitted*

*in fulfillment of the requirements of the degree of Doctor of Philosophy  
to the*



Indian Institute of Technology Delhi  
October 2017

**To**  
**My Parents**

# Certificate

This is to certify that the thesis entitled **Numerical Solution of Some Option Pricing Models** submitted by **Ms. Manisha** to the Indian Institute of Technology Delhi, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by her under my supervision. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for award of any degree or diploma.

New Delhi

October 2017

**Dr. S. Chandra Sekhara Rao**

**Professor**

**Department of Mathematics**

**Indian Institute of Technology Delhi**

**New Delhi-110 016**



# Acknowledgements

*To pursue a career in academics needs perseverance in learning and a mind that is always open for new ideas. My journey as a Research Scholar has taught me the value of endurance and continuance with positive approach. Completing Ph.D. from a renowned institute like Indian Institute of Technology Delhi is the life changing milestone in my career. I am indebted and grateful to many people and institutions for my achievement.*

*First and foremost, I am much obliged to my thesis supervisor Prof. S. Chandra Sekhara Rao. I am extremely blessed to have such an encouraging and helpful supervisor who taught me the essence of doing research. Prof. Rao amazingly eased my journey with his awareness, recognition and understanding of the flourishing research areas. His aptness in anticipating my academic as well as non academic vacillations and ups and downs always kept me proceeding in the right direction. His knowledge, passion, motivation and patience were the crucial element in accomplishment of this thesis.*

*I am also highly grateful to Prof. R. K. Sharma, for his kind cooperation and support in the capacity: as a Chairperson SRC (Student Research Committee). I would like to give my special gratitude to my other SRC members Prof. Rajesh Khanna and Prof. Ritumoni Sarma, for their valuable time, comments and suggestions. I am deeply appreciative to all the faculty members of Department of Mathematics IIT Delhi, for their cooperation and support.*

*I sincerely acknowledge the efforts of the examiners/reviewers. It has greatly helped in improving the thesis.*

*I am extremely thankful to the authorities of IIT Delhi for all the essential facilities provided to me*

throughout the Ph.D. program. I would like to express my sincere appreciation to NBHM (National Board for Higher Mathematics), Department of Atomic Energy, Government of India for providing me research fellowship.

I am also obliged to my seniors Dr. Sunil Kumar and Dr. Mukesh Kumar for their guidance. Many thanks goes to my friends Kavita and Sweta for their love and support. I have shared nice memories with lots of friends including Arti, Amita, Anju, Sudhakar, Swati and Prachi during the journey. I really appreciate the encouragement given by my friends Lokpati, Pragya and Neha.

I am immensely indebted to my Mummy and Papa for their unending support, encouragement and patience. No words can express my gratitude towards them. My special appreciation goes to my brother (Ankit) and my sisters (Anuradha, Deepshikha and Nimisha) for their affection, assistance and sacrifices. I pay my regards to my brothers-in-law (Mr. Amit Jog and CA R. K. Srivastava) for inspiring and motivating me throughout the journey.

I am greatly thankful to my new family (especially my Mummy ji, Papa ji and Baba ji) for their understanding, inspiration, patience and blessings. I accredit my brothers-in-law (Mr. Ankit Kumar and Mr. Raj S Jha) and my sisters-in-law (Mrs. Varsha Sinha and Mrs. Astha Sinha) for their appreciating words. I feel truly fortunate to have my husband (Mr. Udit Kumar) who believes in me and motivates me to reach new heights.

Above all I am eternally indebted to God for graciously bestowing his blessings and making this happen.

New Delhi

Manisha

# Abstract

Some of the European style option pricing models of the financial market are in the forms of: one dimensional degenerate parabolic partial differential equation, ultra-parabolic partial differential equation, partial integro-differential equation and two dimensional parabolic partial differential equation. When the real market complexities are taken into account, the closed form solutions of these models are not available. We develop some high-order convergent numerical methods for such problems.

A high-order difference approximation with identity expansion (HODIE) scheme along with two-step backward differentiation formula is applied to the generalized Black-Scholes option pricing model for four different European option styles which are degenerate parabolic partial differential equation problems. It is proved that the method has second order convergence in space as well as in time. This scheme is also applied to solve dimension reduced generalized Black-Scholes model for pricing Asian option. Further this scheme is extended to produce an accelerated convergence in space when applied to the log transformed generalized Black-Scholes partial differential equation problem.

The HODIE scheme along with Simpson's  $\frac{1}{3}$ rd formula and two-step backward differentiation formula, altogether as an implicit explicit scheme, are applied to solve generalized case of jump-diffusion model for pricing European put option numerically. The method has third order of accuracy in space and second order accuracy in time.

The Peaceman-Rachford Alternating Direction Implicit scheme for temporal semi-discretization and the HODIE scheme for spacial semi-discretization is proposed for numerically solving the two asset Black-Scholes model that is transformed into two dimensional diffusion equation, and the method is proved to be second order accurate in time and fourth order accurate in space.

A first order accurate Alternating Direction Implicit scheme, which is a two dimensional analogue to the backward Euler formula is developed and applied along with the HODIE scheme for solving the generalized model for Asian option which is governed by an ultra-parabolic two dimensional partial differential equation problem. The scheme is proved to be second order accurate in space.

Numerical experiments are in support of the theoretical results and demonstrate the effectiveness of the designed numerical methods.

## सारांश

वित्तीय बाज़ार में यूरोपीय शैली के ऑप्शन मूल्य निर्धारण तंत्रों में से कुछ निम्न रूप में हैं : एक आयामी अपकृष्ट परवलयिक आंशिक अवकल समीकरण, अल्ट्रा-परवलयिक आंशिक अवकल समीकरण, आंशिक समाकल-अवकल समीकरण तथा दो आयामी परवलयिक आंशिक अवकल समीकरण। जब वास्तविक बाज़ार की जटिलताओं को समाविष्ट किया जाता है, तब इन तंत्रों का सटीक हल उपलब्ध नहीं होता। हमने इस प्रकार की समस्याओं के लिए कुछ उच्च ऑर्डर की अभिसृत संख्यात्मक विधियां विकसित की हैं।

चार विभिन्न यूरोपीय ऑप्शन शैलियों के व्यापक बनाए हुए ब्लैक-शोल्स ऑप्शन मूल्य निर्धारण तंत्र, जो कि अपकृष्ट परवलयिक आंशिक अवकल समीकरण समस्याएं हैं, पर एक हाई-ऑर्डर डिफरेंस अप्प्रोक्सिमेशन विद आइडेंटिटी एक्सपेंशन (HODIE) विधि सह द्वि-पदीय पश्चवर्ती अवकलन सूत्र लगाया गया है। यह प्रमाणित किया जाता है की उपरोक्त विधि का स्थान तथा समय में अभिसरण द्वितीय ऑर्डर का है। एशियाई ऑप्शन के मूल्य निर्धारण के आयाम घटाये हुए एवं व्यापक बनाए हुए ब्लैक-शोल्स तंत्र पर भी यह पद्धति लगाई गई है। तदोपरान्त इस विधि को स्थान में त्वरित अभिसरण बनाने के लिए विस्तारित किया जाता है जब इसे लघुगणक परिणत व्यापक बनाए हुए ब्लैक-शोल्स आंशिक अवकल समीकरण समस्या पर लागू किया जाता है।

सिम्पसन के 1/3 वें फॉर्मूला एवं द्वि-पदीय पश्चवर्ती अवकलन सूत्र के साथ HODIE विधि, सब मिला कर एक अंतर्निहित अपरोक्ष पद्धति के रूप में, यूरोपीय पुट ऑप्शन के मूल्य निर्धारण के लिए व्यापक बनाए हुए जम्प-डिफ्यूज़न तंत्र को संख्यात्मक रूप से हल करने के लिए उपयोग किया जाता है। इस विधि में स्थान में सटीकता का तीसरा ऑर्डर और समय में सटीकता का दूसरा ऑर्डर है।

सामयिक अर्ध-विच्छेदन के लिए पीसमैन-रचफोर्ड एकान्तरिक दिशा अंतर्निहित विधि और स्थानिक अर्द्ध-विच्छेदन के लिए HODIE विधि संख्यात्मक रूप से दो संपत्ति ब्लैक-शोल्स तंत्र को हल करने के लिए प्रस्तावित किया गया है जिसे दो आयामी विसरण समीकरण में रूपांतरित किया गया है, और विधि समय में दूसरे ऑर्डर की

सटीक तथा स्थान में चौथे ऑर्डर की सटीक साबित हुई है ।

एक प्रथम ऑर्डर सटीक एकान्तरिक दिशा अंतर्निहित विधि, जो कि पश्चवर्ती यूलर सूत्र का एक द्वि-आयामी एनालॉग है, को विकसित किया गया है तथा HODIE विधि के साथ एशियाई ऑप्शन के व्यापक बनाए हुए तंत्र, जो की एक अल्ट्रा-परवलयिक द्वि-आयामी आंशिक अवकल समीकरण समस्या है, को हल करने के लिए लगाया गया है ।

संख्यात्मक प्रयोग सैद्धांतिक परिणामों के समर्थन में हैं और रूपांकित संख्यात्मक विधियों की प्रभावशीलता प्रदर्शित करते हैं ।

# Contents

<b>Certificate</b>	<b>i</b>
<b>Acknowledgements</b>	<b>iii</b>
<b>Abstract</b>	<b>v</b>
<b>List of Figures</b>	<b>xi</b>
<b>List of Tables</b>	<b>xiii</b>
<b>Notations</b>	<b>xv</b>
<b>1 Introduction</b>	<b>1</b>
1.1 High Order Difference Approximation with Identity Expansion (HODIE) Scheme . . . . .	7
1.2 Literature Survey . . . . .	8
1.2.1 Generalized Black-Scholes Model . . . . .	9
1.2.2 Generalized Model for Pricing Asian Option . . . . .	10
1.2.3 Jump-Diffusion Model for Option Pricing . . . . .	10
1.2.4 Black-Scholes Model for Two-Asset Option Pricing . . . . .	11
1.2.5 Generalized Asian Option Pricing Model in Two Dimensional Form	12
1.3 Organization of the Thesis . . . . .	12

---

<b>2</b>	<b>Generalized Black-Scholes Model for Different Option Styles</b>	<b>15</b>
2.1	Some Modifications in the Model and Localization . . . . .	18
2.2	Discretization . . . . .	25
2.3	Error Analysis . . . . .	29
2.4	Numerical Experiments . . . . .	33
2.5	Conclusions . . . . .	45
<b>3</b>	<b>On Generalized Black-Scholes Model for European Call Option</b>	<b>49</b>
3.1	Some Modifications in the Model and Localization . . . . .	50
3.2	Discretization . . . . .	52
3.3	Error Analysis . . . . .	56
3.4	Numerical Experiments . . . . .	58
3.5	Conclusions . . . . .	60
<b>4</b>	<b>Generalized Model for Pricing Asian Option</b>	<b>63</b>
4.1	Transformations . . . . .	65
4.2	Discretization . . . . .	68
4.3	Error Analysis . . . . .	70
4.4	Numerical Experiments . . . . .	73
4.5	Conclusions . . . . .	75
<b>5</b>	<b>Jump-Diffusion Model for Option Pricing</b>	<b>77</b>
5.1	Transformation and Localization . . . . .	81
5.2	Discretization . . . . .	84
5.3	Error Analysis . . . . .	89
5.4	Numerical Experiments . . . . .	91
5.5	Conclusions . . . . .	92
<b>6</b>	<b>Black-Scholes Model for Two-Asset Option Pricing</b>	<b>95</b>
6.1	Transformations and Localization . . . . .	97
6.2	Time Semidiscretization . . . . .	99

---

6.3	Spatial Semidiscretization . . . . .	101
6.4	Total Discretization . . . . .	105
6.5	Numerical Experiments . . . . .	108
6.6	Conclusions . . . . .	110
<b>7</b>	<b>Generalized Asian Option Pricing Model in Two-Dimensional Form</b>	<b>113</b>
7.1	Transformation and Localization . . . . .	115
7.2	Time Semidiscretization . . . . .	116
7.3	Spatial Semidiscretization . . . . .	119
7.4	Total Discretization . . . . .	125
7.5	Numerical Experiments . . . . .	126
7.6	Conclusions . . . . .	130
	<b>References</b>	<b>133</b>
	<b>Bio-Data</b>	<b>143</b>



# List of Figures

2.1.1 Payoff diagrams . . . . .	21
2.4.1 . . . . .	35
2.4.2 . . . . .	36
2.4.3 Numerical solution of Example 2.4.2(a) . . . . .	37
2.4.4 Numerical solution of Example 2.4.2(b) . . . . .	38
2.4.5 Numerical solution of Example 2.4.3(a) . . . . .	39
2.4.6 Numerical solution of Example 2.4.3(b) . . . . .	40
2.4.7 Numerical solution of Example 2.4.4(a) . . . . .	41
2.4.8 Numerical solution of Example 2.4.4(b) . . . . .	42
2.4.9 Numerical solution of Example 2.4.5(a) . . . . .	43
2.4.10 Numerical solution of Example 2.4.5(b) . . . . .	44
2.4.11 Numerical solution of Example 2.4.5(c) . . . . .	45
3.4.1 Computed solution for Example 3.4.1 . . . . .	60
3.4.2 Computed solution for Example 3.4.2 . . . . .	60
4.4.1 Computed solution of Example 4.4.1 . . . . .	75
4.4.2 Computed solution for Asian option price . . . . .	75
4.4.3 Computed solution for Asian option price . . . . .	76

5.4.1 Numerical solution of European put option under Merton jump-diffusion model . . . . .	92
6.5.1 . . . . .	108
6.5.2 . . . . .	109
6.5.3 . . . . .	110
6.5.4 . . . . .	110
7.5.1 Computed solution for Example 7.5.1 . . . . .	130
7.5.2 Computed solution for Example 7.5.1 . . . . .	130

# List of Tables

2.4.1	Maximum absolute error ( $\hat{E}_{\max}$ ), root mean square error ( $\hat{E}_{\text{rms}}$ ) and corresponding orders of convergence $\hat{p}_{\max}$ and $\hat{p}_{\text{rms}}$ for Example 2.4.1(a) . . .	35
2.4.2	Maximum absolute error ( $\hat{E}_{\max}$ ), root mean square error ( $\hat{E}_{\text{rms}}$ ) and corresponding orders of convergence $\hat{p}_{\max}$ and $\hat{p}_{\text{rms}}$ for Example 2.4.1(b) . . .	36
2.4.3	Maximum absolute error ( $E_{\max}$ ), root mean square error ( $E_{\text{rms}}$ ) and corresponding orders of convergence $p_{\max}$ and $p_{\text{rms}}$ for Example 2.4.2(a) . . .	37
2.4.4	Maximum absolute error ( $E_{\max}$ ), root mean square error ( $E_{\text{rms}}$ ) and corresponding orders of convergence $p_{\max}$ and $p_{\text{rms}}$ for Example 2.4.2(b) . . .	38
2.4.5	Maximum absolute error ( $E_{\max}$ ), root mean square error ( $E_{\text{rms}}$ ) and corresponding orders of convergence $p_{\max}$ and $p_{\text{rms}}$ for Example 2.4.3(a) . . .	39
2.4.6	Maximum absolute error ( $E_{\max}$ ), root mean square error ( $E_{\text{rms}}$ ) and corresponding orders of convergence $p_{\max}$ and $p_{\text{rms}}$ for Example 2.4.3(b) . . .	40
2.4.7	Maximum absolute error ( $E_{\max}$ ), root mean square error ( $E_{\text{rms}}$ ) and corresponding orders of convergence $p_{\max}$ and $p_{\text{rms}}$ for Example 2.4.4(a) . . .	42
2.4.8	Maximum absolute error ( $E_{\max}$ ), root mean square error ( $E_{\text{rms}}$ ) and corresponding orders of convergence $p_{\max}$ and $p_{\text{rms}}$ for Example 2.4.4(b) . . .	43
2.4.9	Maximum absolute error ( $E_{\max}$ ), root mean square error ( $E_{\text{rms}}$ ) and corresponding orders of convergence $p_{\max}$ and $p_{\text{rms}}$ for Example 2.4.5(a) . . .	44

---

2.4.10	Maximum absolute error ( $E_{\max}$ ), root mean square error ( $E_{\text{rms}}$ ) and corresponding orders of convergence $p_{\max}$ and $p_{\text{rms}}$ for Example 2.4.5(b) . . .	45
2.4.11	Maximum absolute error ( $E_{\max}$ ), root mean square error ( $E_{\text{rms}}$ ) and corresponding orders of convergence $p_{\max}$ and $p_{\text{rms}}$ for Example 2.4.5(c) . . .	46
3.4.1	Maximum absolute error ( $E_{\max}$ ), root mean square error ( $E_{\text{rms}}$ ) and corresponding orders of convergence $p_{\max}$ and $p_{\text{rms}}$ for Example 3.4.1 . . . .	59
3.4.2	Maximum absolute error ( $E_{\max}$ ), root mean square error ( $E_{\text{rms}}$ ) and corresponding orders of convergence $p_{\max}$ and $p_{\text{rms}}$ for Example 3.4.2 . . . .	59
4.4.1	Maximum absolute error ( $E_{\max}$ ), root mean square error ( $E_{\text{rms}}$ ) and corresponding orders of convergence $p_{\max}$ and $p_{\text{rms}}$ for Example 4.4.1 . . . .	74
4.4.2	Maximum absolute error ( $E_{\max}$ ), root mean square error ( $E_{\text{rms}}$ ) and corresponding orders of convergence $p_{\max}$ and $p_{\text{rms}}$ for Example 4.4.2 . . . .	74
4.4.3	Maximum absolute error ( $E_{\max}$ ), root mean square error ( $E_{\text{rms}}$ ) and corresponding orders of convergence $p_{\max}$ and $p_{\text{rms}}$ for Example 4.4.3 . . . .	74
5.4.1	Maximum absolute error ( $E_{\max}$ ), root mean square error ( $E_{\text{rms}}$ ) and corresponding orders of convergence $p_{\max}$ and $p_{\text{rms}}$ for Example 5.4.1 . . . .	91
7.5.1	Maximum absolute error ( $E_{\max}$ ), root mean square error ( $E_{\text{rms}}$ ) and corresponding orders of convergence $p_{\max}$ and $p_{\text{rms}}$ for Example 7.5.1 . . . .	128
7.5.2	Maximum absolute error ( $E_{\max}$ ), root mean square error ( $E_{\text{rms}}$ ) and corresponding orders of convergence $p_{\max}$ and $p_{\text{rms}}$ for Example 7.5.2 . . . .	129

# Notations

$S$	asset price
$S_0$	initial asset price
$\tau, t$	time variables
$C$	call option price
$P$	put option price
$\sigma$	market volatility
$r$	risk free interest rate
$D$	dividend yield
$K$	strike price
$T$	maturity time of the contract
$W$	Wiener process
$\varepsilon$	small parameter
$a_2$	diffusion coefficient
$a_1$	convection coefficient
$a_0$	reaction coefficient
$\alpha_i, i = -, c, +$	HODIE coefficients corresponding to the stencil points
$\beta_i, i = -, c, +$	HODIE coefficients corresponding to the auxiliary points
$h$	grid length in space direction
$k$	grid length in time direction
$M, N$	mesh discretization parameters

---

$L$	continuous differential operator
$\Omega_S, \Omega_x$	space variable domain
$\Omega_t$	variable domain
$\Omega$	the cross product of space and time domains
$E_{\max}$	maximum absolute error
$E_{\text{rms}}$	root mean square error
$\ \cdot\ $	norm when domain is obvious, or of no particular significance
$v_i$	$v(x_i)$
$\mathcal{C}, \mathcal{C}_s, s = 0, 1, \dots$	generic positive constants, independent of $M, N$