

The σ -Compact-Open Topology on $C^*(X)$ and the Corresponding Dual Space

By

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Certificate

This is to certify that the thesis entitled *The σ -Compact-Open Topology on $C^*(X)$ and the Corresponding Dual Space* submitted by *Mr Vipra Pandey* to the Indian Institute of Technology Delhi, for the award of the Degree of Doctor of Philosophy, is a record of the original bona fide research work carried out by him under my guidance and supervision. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi

February 2011

Dr. Subiman Kundu

Supervisor

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Abstract

This thesis studies the σ -compact-open topology on $C^*(X)$, the set of all bounded, continuous real-valued functions defined on a Tychonoff space X and the corresponding dual space.

The space $C_\sigma^*(X)$ denotes the topological space $C^*(X)$ equipped with the σ -compact-open topology. After having a few basic results on $C_\sigma^*(X)$, we compare the σ -compact-open topology on $C^*(X)$ with the well-known topologies on $C^*(X)$ such as the compact-open topology, the strict topology and the topology of uniform convergence on $C^*(X)$. Then we study the following topological properties of $C_\sigma^*(X)$ in terms of the topological properties of X :

- (i) submetrizability and metrizability,
- (ii) various kinds of completeness properties such as uniform completeness, quasi completeness, sequential completeness, complete metrizability, Čech-completeness, almost Čech-completeness, local Čech-completeness, sieve-completeness, partition-completeness, pseudocompleteness and the property of being a Baire space, and
- (iii) various countability properties such as separability, countable chain condition, second countability and Lindelöf property.

We also study nuclear and certain other related properties of $C_\sigma^*(X)$.

Since $C_\sigma^*(X)$ is a locally convex space, naturally this thesis also studies its dual space, that is, the space of all continuous linear functionals on $C_\sigma^*(X)$. This dual space is denoted by $\Lambda_\sigma(X)$ which inherits a natural norm from a larger Banach space. We study the natural question of the completeness of this norm on $\Lambda_\sigma(X)$ as well as a compatible lattice structure on it. Since continuous functions and Baire measures on Tychonoff spaces are linked by the process of integration, it is very natural to identify an element of $\Lambda_\sigma(X)$ with a suitable measure on X . So finally we study the measure-theoretic characterization of $\Lambda_\sigma(X)$.

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