EXTREMAL PROPERTIES AND COEFFICIENT ESTIMATES FOR POLYNOMIALS WITH RESTRICTED ZEROS AND ON LOCATION OF ZEROS OF POLYNOMIALS

KUM KUM DEWAN

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Department of Mathematics
Indian Institute of Technology,Delhi
Hauz Khas, New Delhi-110029.

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SYNOPSIS

This thesis consists of three chapters. The first chapter deals with the extremal properties of a polynomial. In the second chapter we have studied the problems concerning the location of zeros of a polynomial and in the third chapter some coefficient problems for polynomials have been considered.

If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n, then according to Bernstein's theorem

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|, \tag{1}$$

$$\max_{|z|=R\geq 1} |p(z)| \leq R^n \max_{|z|=1} |p(z)|, \tag{2}$$

$$\int_0^{2\pi} |p'(e^{i\theta})|^2 \, d\theta \leq n^2 \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta \tag{3}$$

and

$$\int_0^{2\pi} |p(Re^{i\theta})|^2 \, d\theta \leq R^{2n} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta \tag{4}$$

Lax [48] and Ankeny and Rivlin [3] considered the class of polynomials p(z) having no zeros inside the unit circle and obtained inequalities analogous to (1) and (2). Rahman [67] considered the class of polynomials having no zero in $|z| < K$, $K \leq 1$ and obtained the inequalities analogous to (3) and (4). The class of polynomials having no zero in $|z| < K$, $K \geq 1$ was studied by Malik [49]. (Also see Govil and Rahman [35]). Govil, Jain and Labelle [33] considered the class of polynomials satisfying $p(z) \equiv z^n\, p(\frac{1}{z})$ and having the zeros either in the left half plane or in the right half plane while O'hara and Rodriguez [60] considered the class of polynomials satisfying $p(z) \equiv z^n\, \overline{p(\frac{1}{\bar{z}})}$. In the first chapter we consider the class of polynomials satisfying $p(z) \equiv z^n\, p(\frac{1}{z})$ and prove a result of Govil, Jain and Labelle [33] without the condition that $p(z)$ has all its zeros in one half plane. Our result is best possible and generalizes the result due to Govil, Jain and Labelle [33]. Also we obtain L^2 inequalities for the sth derivative of the classes of polynomials having no zero in $|z| < K$, $K \leq 1$ and for polynomials having no zero in $|z| < K$, $K \geq 1$. These results generalize some well known results. The inequality for the case $K \leq 1$, is best possible. For the class of polynomials having no zeros in $|z| < K$, $K \leq 1$ we as well obtain inequality analogous to (2). Besides some other results for these classes and other related classes have also been obtained.

A classical result of Cauchy on the location of the zeros of the polynomial $p(z) = z^n + \sum_{\nu=0}^{n-1} a_\nu z^\nu$ states that all the zeros are in the circle $|z| \leq 1+A$ where $A = \max_{0 \leq j < n} |a_j|$. Joyal, Labelle and Rahman [42] and

Brham Datt and Govil [18] improved the above mentioned result of Cauchy. Further various generalizations of Eneström-Kakeya Theorem have been obtained, among others by Cargo and Shisha [13a], Joyal, Labelle and Rahman [42], P.V. Krishnaiah [46], Rubinstein [75], Govil and Rahman [34] and Govil and Jain [32]. In the second chapter we obtain an improvement of the result of Joyal, Labelle and Rahman [42] and Brham Datt and Govil [18] dealing with Cauchy's theorem by obtaining the zero free region bigger than the zero free region obtained by them. We also sharpen the result proved by Joyal, Labelle and Rahman [42] and Govil and Jain [32] concerning Eneström-Kakeya theorem by obtaining a smaller region. Besides these, various other results concerning the location of zeros of polynomials have been obtained.

Let $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n. If $M = \max_{|z|=1} |p(z)|$, then, by Cauchy's inequality, we have

$$|a_\nu| \leqslant M, \quad 0 \leqslant \nu \leqslant n. \tag{5}$$

Also, by a very simple method C.Visser [86] proved that

$$|a_0| + |a_n| \leqslant M \tag{6}$$

Rahman [66] gave the estimate for the sum of the moduli of any two coefficients of a polynomial p(z) in terms of $M = \max_{|z|=1} |p(z)|$ and proved that for $0 \leqslant u < V \leqslant n$

$$|a_u| + |a_v| \leqslant \frac{4M}{\pi} \tag{7}$$

O'hara and Rodriguez [60] considered the class of polynomials having all its zeros on the unit circle. The class of polynomials $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ having a zero on $|z| = \rho$, $(0 < \rho < \infty)$ was studied by Rahman and Schmeisser [70] and they obtained a sharp upper bound for $|a_0|$ in terms of $\max_{|z|=1} |p(z)|$. In the third chapter we consider the class of polynomials having a prescribed zero on $|z| = 1$ and obtain an inequality analogous to (5). We also obtain inequalities analogous to (7) for the class of polynomials having all its zeros on $|z| = 1$ and for the class of polynomials whose zeros lie on or exterior (interior) to $|z| = 1$. Besides these, some other related results have also been obtained.

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