SOME EXISTENCE AND MULTIPLICITY RESULTS FOR ELLIPTIC EQUATIONS WITH DEGENERATIONS AND SINGULARITIES

BY

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Certificate

I am satisfied that the thesis entitled *Some Existence And Multiplicity Results for Elliptic Equations with degenerations and singularities* presented by Ms. Sumit Kaur Bhatia (2006MAZ8087) is worthy of consideration for the award of the degree of Doctor of philosophy and is a record of the original bonafide research work carried out by her under my guidance and supervision and the results contained in it have not been submitted in part or full to any other university or Institute for award of any degree/diploma.

New Delhi **Dr. K. Sreenadh**

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Abstract

In this Thesis, we study the existence, nonexistence and multiplicity of positive solutions of a class of quasilinear degenerate equations with critical and singular nonlinearities in bounded domains.

First chapter of thesis consists of brief survey and preliminary results we require in subsequent chapters.

In the second chapter, we consider the following conormal derivative problem for $p-$ Laplacian equation on a bounded domain $\Omega \subset I\!R^N$.

$$\begin{aligned} -\Delta_p u + u^{p-1} &= u^{p^*-1}, \quad u > 0 \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= \lambda u^q \ \text{ on } \ \partial\Omega. \end{aligned}$$

where $\lambda > 0, p^* = \frac{Np}{N-p}, 0 \leq q < p-1$ and $\frac{\partial u}{\partial \nu}$ is the normal derivative of u.

This problem is motivated from the works of Ambrosetti, Brezis and Cerami [8], Garcia, Peral and Rossi [36], where existence of two positive solutions for maximal range of $\lambda \in (0, \Lambda)$ was obtained. While the results of [8] were generalized to $p-$ Laplacian equation in [25], the multiplicity for conormal problem as in [36] was left open. In this chapter, we study this problem.

In the third chapter, we study the uniform L^∞ estimates for elliptic equations with critical nonlinearities and obtain an important property that C^1 local minimum of a class of elliptic functionals is also $W^{1,N}$ local minimum. This property is found to be important for obtaining global multiplicity results.

In the fourth chapter, we study the existence and multiplicity for $\phi-$ Laplacian equation having exponential growth of critical type. We consider the problem:

$$\begin{aligned} -div(\phi(|\nabla u|)\nabla u) + \phi(|u|)u &= f(u) \;\; u > 0 \quad \text{in } \Omega, \\ \phi(|\nabla u|)\frac{\partial u}{\partial \nu} &= \lambda u^q \;\text{ on }\; \partial\Omega, \end{aligned}$$

where ϕ is a C^1 function such that $\phi(t) \sim t^{p_0-2}$ as $t \to 0, \phi(t) \sim t^{p_1-2}$ as $t \to \infty$ and $f(t) \sim t^p e^{t^{N/N-1}}$. We generalize the results in the second chapter to this case.

In the fifth chapter, we consider singular problem motivated from Hardy-Sobolev inequality.

$$\left\{ \begin{array}{rcl} \left.\begin{array}{rcl} -\Delta u + u &=& h(x,u)\frac{e^{u^2}}{|x|^\beta} \\ u &>& 0 \end{array}\right\} & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda \psi u^q \;\text{ on }\; \partial\Omega, & \end{array} \right.$$

where $0 \in \partial\Omega$, $\beta \in [0,2), \lambda > 0, q \in [0,1)$ and $\psi \geq 0$, a Hölder continuous function on $\overline{\Omega}$. Here $h(x,u)$ is a $C^1(\overline{\Omega} \times I\!R)$ having superliner growth at infinity. Using variational methods we show that there exists $0 < \Lambda < \infty$ such that the above problem admits atleast two solutions in $H^1(\Omega)$ if $\lambda \in (0,\Lambda)$, no solution if $\lambda > \Lambda$ and atleast one solution when $\lambda = \Lambda$. Here solutions are singular at 0.

In the sixth chapter, we study the existence of multiple positive solutions of singular Biharmonic problem in four dimensions with exponential nonlinearity and with Dirichlet boundary condition. Let $B_1(0) \subset I\!R^4$ be the unit ball.

$$\left\{ \begin{aligned} \Delta^2 u &= \frac{h(u)}{|x|^\beta} e^{u^2} + \lambda u^q, \;\; u > 0 \quad \text{in } B_1, \\ u &= \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial B_1, \end{aligned} \right.$$

where $0 < q < 1, \lambda \in I\!R, 0 < \beta < 4$. We show that there exists $0 < \Lambda < \infty$ such that the above problem admits atleast two solutions in $H_0^2(B_1)$ if $\lambda \in (0,\Lambda)$, no solution if $\lambda > \Lambda$ and atleast one solution when $\lambda = \Lambda$.

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