

WAVELET COLLOCATION METHODS FOR FRACTIONAL OPTIMAL CONTROL PROBLEMS

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by

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*Dedicated to
my Parents*

Certificate

This is to certify that the thesis entitled **Wavelet Collocation Methods for Fractional Optimal Control Problems** submitted by **Mr. Nitin Kumar** to the **Indian Institute of Technology Delhi**, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi
July 2023

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Abstract

Fractional calculus has found many applications in mathematical physics and engineering. In recent years, the development of fractional calculus was rapidly increased. This is because the realistic modeling of a physical phenomenon that has a dependence on both the time instance and the previous time history can be successfully achieved with fractional calculus. On the other hand, fractional differential equations and fractional partial differential equations provide more accurate models of many engineering systems due to their non-local properties and describe precisely the dynamic behavior of the systems in various sciences than ordinary differential equations and partial differential equations, respectively.

This thesis is concerned with the numerical methods and analysis of the fractional optimal control problems. Throughout the thesis, we have predominantly used wavelet collocation methods to solve fractional optimal control problems. There are various polynomial-generated wavelets in the Literature, for example, Hermite wavelet, Legendre wavelet, Chebyshev wavelet, etc. We started with Hermite wavelets, but later we realized that each polynomial-generated wavelet gives the same order of the error in function approximation. So, we have adopted Legendre wavelet or its more generalized form in the later chapters. Moreover, we have considered fractional optimal control problems in one-dimensional and two-dimensional cases. First, we propose numerical methods for solving fractional optimal control problems in the one-dimensional case, then we move to the two-dimensional case.

We started by reviewing some literature and known results on fractional calculus, which plays a crucial role in the following chapters. First, we present an efficient numerical method for solving fractional optimal control problems (FOCPs) using the Hermite scaling function operational matrix of fractional order integration. The proposed technique is applied to transform the state and control variables into non-linear programming (NLP) parameters at collocation points.

The NLP solver is then used to solve FOCP. Furthermore, the L_2 -error estimates in the approximation of unknown variables by Hermite wavelet and the approximation of block pulse operational matrix of fractional order integration are derived, and illustrative examples are included to demonstrate the applicability of the proposed method. Moreover, we have also compared the results with some existing methods as the Haar wavelet collocation method, the hybrid of block-pulse and Taylor polynomials method, the Bernstein polynomials method, and the Boubaker hybrid function method to show the superiority of the proposed method.

In the case of a fractional optimal control problem with fractional cost, we consider two problems: first is a general fractional optimal control problem (FOCP) involving a dynamical system described by a nonlinear Caputo fractional differential equation, associated with a fractional Bolza cost composed as the aggregate of a standard Mayer cost and a fractional Lagrange cost given by a Riemann-Liouville fractional integral. The second one is its extension to the variable order. Using the Lagrange multiplier within the calculus of variations and applying integration by part formula, the necessary optimality conditions are derived in terms of a nonlinear two-point boundary value problem. An operational matrix of fractional order right Riemann-Liouville integration is proposed. By utilizing it, the obtained two-point fractional-order boundary value problem is reduced into the solution of an algebraic system. An L_2 -error estimate in the approximation of unknown variable by Legendre wavelet is derived, and illustrative examples are included to demonstrate the applicability of the proposed method.

We also consider a fractional optimal control problem with dynamic constraint as a fractional Sturm-Liouville problem and develop a Müntz-Legendre wavelet method for solving this problem. We first derive the necessary optimality conditions as a two-point boundary value problem using the calculus of variation and integration by part formula. The operational matrices for Müntz-Legendre scaling functions have been obtained and by utilizing it, the two-point boundary value problem has been converted into a system of algebraic equations. Then L_2 -error estimates in the approximation of operations matrices and in the approximation of unknown variables by the Müntz-Legendre wavelet have been derived. Illustrative examples have been taken to show the applicability of the proposed method.

Now, we move to fractional optimal control problems driven by fractional partial differential equations, called two-dimensional fractional optimal control problems (FOCPs-2D). We first present the necessary optimality conditions and a new method for solving a class of two-dimensional fractional optimal control problems based on shifted Legendre polynomials. We

utilize the Lagrange multiplier method and integration by part formula within the calculus of variations to derive the necessary optimality conditions as a two-point fractional-order boundary value problem. Fractional operators of shifted Legendre polynomials are computed and the necessary optimality conditions have been converted into a system of algebraic equations by utilizing these fractional operators. L_2 -error estimates in the approximation of a function and its fractional derivative by shifted Legendre polynomial have been derived. Moreover, convergence analysis of the proposed method has also been discussed. Illustrative examples are included to demonstrate the effectiveness and applicability of the proposed method.

We have presented the necessary optimality conditions and a new method for solving a class of two-dimension Sturm-Liouville fractional optimal control problems based on the generalized fractional-order Legendre scaling function. The Lagrange multiplier method and integration by part formula within the calculus of variations have been utilized to derive the necessary optimality conditions as a two-point fractional-order boundary value problem. The operational matrix for generalized fractional-order Legendre scaling functions has been derived, and the necessary optimality conditions have been converted into the system of algebraic equations by utilizing these operational matrices. Moreover, the L_2 -error estimate in the approximation of a function by generalized fractional-order Legendre scaling function has been derived. Illustrative examples have been taken to demonstrate the performance and applicability of the proposed method.

Finally, we have proposed a numerical method based on the generalized fractional-order Legendre wavelet for solving fractional optimal control problems with constraints as two-dimensional distributed order fractional differential equations. An exact formula for the Riemann-Liouville integration of generalized fractional-order Legendre wavelet has been derived by using regularized beta functions. This formula and the two-dimensional Gauss-Legendre integration formula have been used to solve the two-dimensional distributed order fractional optimal control problem. Moreover, an L_2 -error estimate in the approximation of an unknown function with the generalized fractional-order Legendre wavelet has been derived, and the estimated order has been verified for a test function. Furthermore, convergence analysis for the proposed method has been presented. Two test problems have been considered to illustrate the efficiency of the proposed method.

सार

फ्रैक्शनल कैलकुलस को गणितीय भौतिकी और इंजीनियरिंग में कई अनुप्रयोग मिले हैं। हाल के वर्षों में, फ्रैक्शनल कैलकुलस का विकास तेजी से बढ़ा है। ऐसा इसलिए है क्योंकि एक भौतिक घटना का यथार्थवादी मॉडलिंग जो समय के उदाहरण और पिछले समय के इतिहास दोनों पर निर्भर करता है, उसे भिन्नात्मक कलन के साथ सफलतापूर्वक प्राप्त किया जा सकता है। दूसरी ओर, आंशिक अंतर समीकरण और आंशिक आंशिक अंतर समीकरण अपने गैर-स्थानीय गुणों के कारण कई इंजीनियरिंग प्रणालियों के अधिक सटीक मॉडल प्रदान करते हैं। और क्रमशः सामान्य अंतर समीकरणों और आंशिक अंतर समीकरणों की तुलना में विभिन्न विज्ञानों में प्रणालियों के गतिशील व्यवहार का सटीक वर्णन करते हैं।

यह थीसिस आंशिक इष्टतम नियंत्रण समस्याओं के संख्यात्मक तरीकों और विश्लेषण से संबंधित है। संपूर्ण थीसिस में, हमने मुख्य रूप से तरंगिका संयोजन विधियों का उपयोग किया है। भिन्नात्मक इष्टतम नियंत्रण समस्याओं को हल करने के लिए साहित्य में विभिन्न बहुपद-जनित तरंगिकाएँ हैं, उदाहरण के लिए, हर्माइट तरंगिका, लीजेंड्रे तरंगिका, चेबीशेव तरंगिका, आदि। शुरुआत हर्माइट वेवलेट्स से हुई, लेकिन बाद में हमें एहसास हुआ कि प्रत्येक बहुपद-जनित वेवलेट फ्रैक्शनल सन्निकटन में त्रुटि का समान क्रम देता है। इसलिए, हमने लीजेंड्रे को अपनाया है। बाद के अध्यायों में वेवलेट या इसका अधिक सामान्यीकृत रूप। इसके अलावा, हमने एक-आयामी और दो-आयामी मामलों में भिन्नात्मक इष्टतम नियंत्रण समस्याओं पर विचार किया है। सबसे पहले, हम समर्थक- एक-आयामी मामले में भिन्नात्मक इष्टतम नियंत्रण समस्याओं को हल करने के लिए संख्यात्मक तरीके प्रस्तुत करें, फिर हम द्वि-आयामी मामले की ओर बढ़ते हैं।

हमने भिन्नात्मक कैलकुलस पर कुछ साहित्य और ज्ञात परिणामों की समीक्षा करके शुरुआत की, जो निम्नलिखित अध्यायों में महत्वपूर्ण भूमिका निभाता है। सबसे पहले, हम हल करने के लिए एक कुशल संख्यात्मक विधि प्रस्तुत करते हैं।

भिन्नात्मक क्रम एकीकरण के हार्माइट स्केलिंग फंक्शन परिचालन मैट्रिक्स का उपयोग करके भिन्नात्मक इष्टतम नियंत्रण समस्याएं (एफओसीपी)। प्रस्तावित तकनीक को कोलोकेशन बिंदुओं पर स्थिति और नियंत्रण चर को गैर-रेखीय प्रोग्रामिंग (एनएलपी) मापदंडों में बदलने के लिए लागू किया जाता है। फिर एनएलपी सॉल्वर का उपयोग एफओसीपी को हल करने के लिए किया जाता है। इसके अलावा, हर्मिट वेवलेट द्वारा अज्ञात चर के सन्निकटन और ब्लॉक पल्स के सन्निकटन में एल2-त्रुटि का अनुमान है। भिन्नात्मक क्रम एकीकरण के परिचालन मैट्रिक्स प्राप्त किए गए हैं, और प्रस्तावित विधि की प्रयोज्यता को प्रदर्शित करने के लिए उदाहरणात्मक उदाहरण शामिल किए गए हैं। इसके अलावा, हमने प्रस्तावित विधि की श्रेष्ठता दिखाने के लिए कुछ मौजूदा तरीकों जैसे हार वेवलेट कोलोकेशन विधि, ब्लॉक-पल्स और टेलर बहुपद विधि, बर्नस्टीन बहुपद विधि और बाउबेकर हाइब्रिड फंक्शन विधि के साथ परिणामों की तुलना भी की है।

भिन्नात्मक लागत के साथ भिन्नात्मक इष्टतम नियंत्रण समस्या के मामले में, हम दो समस्याओं पर विचार करते हैं। पहला एक सामान्य भिन्नात्मक इष्टतम नियंत्रण समस्या (एफओसीपी) है जिसमें एक गैर-रेखीय कैप्टो भिन्नात्मक अंतर समीकरण द्वारा वर्णित एक गतिशील प्रणाली शामिल है, जो एक भिन्नात्मक से जुड़ी है। बोलज़ा लागत एक मानक मेयर लागत और एक रीमैन-लिउविले भिन्नात्मक अभिन्न अंग द्वारा दी गई आंशिक लैगेंज लागत के योग के रूप में बनाई गई है। दूसरा इसका परिवर्तनशील क्रम में विस्तार है। विविधताओं की गणना के भीतर लैगेंज गुणक का उपयोग करना और भाग सूत्र द्वारा एकीकरण को लागू करना, एक गैर-रेखीय दो-बिंदु सीमा मूल्य समस्या के संदर्भ में आवश्यक इष्टतमता की स्थिति प्राप्त की जाती है। रीमैन-लिउविले एकीकरण के भिन्नात्मक क्रम का एक परिचालन मैट्रिक्स प्रस्तावित है। इसका उपयोग करके, प्राप्त दो-बिंदु भिन्नात्मक-क्रम सीमा मान समस्या को बीजगणितीय प्रणाली के समाधान में घटा दिया जाता है। लीजेंड्रे वेवलेट द्वारा अज्ञात चर के सन्निकटन में एक एल2-त्रुटि अनुमान प्राप्त किया गया है, और प्रस्तावित विधि की प्रयोज्यता को प्रदर्शित करने के लिए उदाहरणात्मक उदाहरण शामिल किए गए हैं।

हम गतिशील बाधा के साथ एक भिन्नात्मक इष्टतम नियंत्रण समस्या को एक भिन्नात्मक स्टर्म-लिउविले समस्या के रूप में भी मानते हैं। और इस समस्या को हल करने के लिए मंटज़-लीजेंडर वेवलेट विधि विकसित करते हैं। हम पहले भाग सूत्र द्वारा भिन्नता और एकीकरण की गणना का उपयोग करके दो-बिंदु सीमा मूल्य समस्या के रूप में आवश्यक इष्टतमता स्थितियों को प्राप्त करते हैं। मंटज़-लीजेंडर स्केलिंग फंक्शंस के लिए परिचालन मैट्रिक्स प्राप्त कर लिया गया है और इसका उपयोग करके, दो-बिंदु सीमा मान समस्या को बीजगणितीय समीकरणों की एक प्रणाली में परिवर्तित कर दिया गया है। फिर संचालन मैट्रिक्स के सन्निकटन में और मंटज़-लीजेंडर वेवलेट द्वारा अज्ञात चर के सन्निकटन में एल2-त्रुटि अनुमान प्राप्त किया गया है। प्रस्तावित पद्धति की प्रयोज्यता दिखाने के लिए उदाहरणात्मक उदाहरण लिए गए हैं।

अब, हम भिन्नात्मक आंशिक अंतर समीकरणों द्वारा संचालित भिन्नात्मक इष्टतम नियंत्रण समस्याओं की ओर बढ़ते हैं। जिन्हें द्वि-आयामी भिन्नात्मक इष्टतम नियंत्रण समस्याएं (एफओसीपी -2डी) कहा जाता है। हम सबसे पहले स्थानांतरित लीजेंड्रे बहुपदों के आधार पर द्वि-आयामी भिन्नात्मक इष्टतम नियंत्रण समस्याओं के एक वर्ग को हल करने के लिए आवश्यक इष्टतमता की स्थिति और एक नई विधि प्रस्तुत करते हैं। हम कलन के भीतर भाग सूत्र द्वारा लैंग्रेज गुणक विधि और एकीकरण का उपयोग करते हैं। दो-बिंदु भिन्नात्मक-क्रम सीमा मान समस्या के रूप में आवश्यक इष्टतमता स्थितियों को प्राप्त करने के लिए विविधताएं। स्थानांतरित लीजेंड्रे बहुपदों के भिन्नात्मक संचालकों की गणना की गई है और इन भिन्नात्मक संचालकों का उपयोग करके आवश्यक इष्टतमता स्थितियों को बीजगणितीय समीकरणों की एक प्रणाली में परिवर्तित किया गया है। एल2-त्रुटि किसी फंक्शन के सन्निकटन में अनुमान और स्थानांतरित लीजेंड्रे बहुपद द्वारा इसके भिन्नात्मक व्युत्पन्न को प्राप्त किया गया है। इसके अलावा, प्रस्तावित पद्धति के अभिसरण विश्लेषण पर भी चर्चा की गई है। प्रस्तावित पद्धति की प्रभावशीलता और प्रयोज्यता को प्रदर्शित करने के लिए उदाहरणात्मक उदाहरण शामिल किए गए हैं।

हमने सामान्यीकृत भिन्नात्मक-क्रम लीजेंड्रे स्केलिंग फंक्शन के आधार पर दो-आयाम स्टर्म-लिउविले भिन्नात्मक इष्टतम नियंत्रण समस्याओं के एक वर्ग को हल करने के लिए आवश्यक इष्टतमता की स्थिति और एक नई विधि प्रस्तुत की है। दो-बिंदु भिन्नात्मक-क्रम सीमा मूल्य समस्या के रूप में आवश्यक इष्टतमता स्थितियों को प्राप्त करने के लिए लैंग्रेज गुणक विधि और विविधताओं की गणना के भीतर भाग सूत्र द्वारा एकीकरण का उपयोग किया गया है। सामान्यीकृत भिन्नात्मक-क्रम लीजेंड्रे स्केलिंग फंक्शंस के लिए परिचालन मैट्रिक्स प्राप्त किया गया है, और इन परिचालन मैट्रिक्स का उपयोग करके आवश्यक इष्टतमता स्थितियों को बीजगणितीय समीकरणों की प्रणाली में परिवर्तित कर दिया गया है। इसके अलावा, ए के सन्निकटन में एल2-त्रुटि अनुमान सामान्यीकृत भिन्नात्मक-क्रम लीजेंड्रे स्केलिंग फंक्शन द्वारा फंक्शन प्राप्त किया गया है। प्रस्तावित पद्धति के प्रदर्शन और प्रयोज्यता को प्रदर्शित करने के लिए उदाहरणात्मक उदाहरण लिए गए हैं।

अंत में, हमने द्वि-आयामी बाधाओं के साथ भिन्नात्मक इष्टतम नियंत्रण समस्याओं को हल करने के लिए सामान्यीकृत भिन्न-क्रम लीजेंड्रे वेवलेट पर आधारित एक संख्यात्मक विधि प्रस्तावित की है। वितरित क्रम भिन्नात्मक अंतर समीकरण। सामान्यीकृत फ्रैक्शनल-ऑर्डर लीजेंड्रे वेवलेट के रीमैन-लिउविले एकीकरण के लिए एक सटीक सूत्र नियमित बीटा फंक्शंस का उपयोग करके प्राप्त किया गया है। इस सूत्र और द्वि-आयामी गॉस-लीजेंडर एकीकरण सूत्र का उपयोग द्वि-आयामी वितरित क्रम भिन्नात्मक इष्टतम नियंत्रण समस्या को हल करने के लिए किया गया है। इसके अलावा, एक अज्ञात फंक्शन के सन्निकटन में एक एल2-त्रुटि अनुमान सामान्यीकृत भिन्नात्मक-क्रम लीजेंड्रे वेवलेट प्राप्त किया गया है, और अनुमानित क्रम को एक परीक्षण फंक्शन के लिए सत्यापित किया गया है। इसके अलावा, प्रस्तावित पद्धति के लिए अभिसरण विश्लेषण

प्रस्तुत किया गया है। प्रस्तावित पद्धति की दक्षता को दर्शाने के लिए दो परीक्षण समस्याओं पर विचार किया गया है।

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List of Symbols

Symbol	Meaning
\mathbb{N}	Set of natural numbers
\mathbb{R}	Set of real numbers
$A \subseteq X$	A is a subset of X
$A \cap B$	The intersection of A and B
$A \cup B$	The union of A and B
$C[a, b]$	set of all continuous functions
$C^n[a, b]$	Space of all functions having n times continuously differentiable on $[a, b]$
$AC^n[a, b]$	Space of functions f such that $f \in C^{n-1}([a, b])$ and $f^{(n-1)} \in AC([a, b])$

