STOCHASTIC MODELLING AND PRICING OF EQUITY-LINKED ANNUITIES IN LIFE INSURANCE

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Dedicated To My Parents

Certificate

This is to certify that the thesis entitled **"Stochastic Modelling and Pricing of Equity-Linked Annuities in Life Insurance"** submitted by **Mrs. NITU SHARMA** to the Indian Institute of Technology Delhi, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by her under my supervision. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results obtained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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Abstract

Equity-linked annuities are insurance products whose values are linked with some index or a portfolio. These products provide participation in the market with security against downside movement of the market. They have evolved as a perfect investment option over the years.

Equity-linked annuities can be divided into variable annuities and equity-indexed annuities. The difference between these two is that variable annuities enable the policyholder to own an investment in the market whereas in equity-indexed annuities returns are credited to the policyholder's account based on the performance of an underlying index such as S&P 500. Additionally, equity-indexed annuities are moderate risk products, whereas variable annuities are high in risk. The equity-indexed annuities are available in different designs based on their indexing method. A few popular designs of equity-indexed annuities include a point-to-point design and an annual reset design. In a point-to-point design, returns are credited based on the growth between two fixed points whereas, in annual reset design, any gains in a year are credited to policyholders account and are locked in. Similarly, variable annuities come with a living or death benefit rider where living benefit riders provide a guaranteed benefit amount after surviving maturity, and death benefit rider provides a guaranteed benefit amount to the nominee at the instance of death of the policyholder. The living guarantees are similar to pension plans with participation in the market.

This thesis addresses the pricing problem of variable annuities and equity-indexed annuities. This thesis proposes various stochastic models for the pricing of these annuities. The pricing problem of equity-linked annuities majorly involves three types of risks: investment risk, interest rate risk and mortality risk. The equity returns involves various features such as volatility clustering, leverage effect, high kurtosis, etc.. We propose an asymmetric GARCH model to capture these features and develop a pricing model for variable annuities.

During the 2007-08 crisis, many of the variable annuity products became popular and valuable. The drop in the market prices resulted in loss to many of the insurers who were unable to fulfil the living guarantees. One of the reasons for losses was the inaccurate pricing of the variable annuity products. Therefore, this thesis proposes a jump-diffusion model with generalized jump distributions to price living guarantees. Besides, we model the mortality risk by a Gompertz model, and we model the interest rate by Vasicek model. Further, we model some of the additional benefits available to the policyholder, such as surrender option and roll-up option. We use mixed fractional Brownian motion model with jumps to capture the long-range dependency in returns. Additionally, we utilize variations of the popular Lee-Carter model for mortality modelling.

From the literature, it is evident that it is not the initial jump which results in the downturn in the market; it is the amplification of these jumps over time. Therefore, in the end, this thesis proposes a Hawkes jump-diffusion model for the valuation of equity-indexed annuities. Due to the large duration of these products, we model interest rate risk using a Vasicek model. Hawkes jump-diffusion models can capture the phenomena of jump clustering and hence can provide fair pricing of equity-indexed annuities.

सार

इक्विटी-लिंक्ड एन्युटी(Equity-linked annuity) एक बीमा उत्पाद हैं, जिनके मूल्य कुछ सूचकांक या पोर्टफोलियो के साथ जुड़े हुए हैं । ये उत्पाद बाजार में भागीदारी के साथ बाजार के नीचे की ओर जाने से सुरक्षा प्रदान करते हैं।
पिछले कुछ वर्षों में ये एक संपूर्ण निवेश विकल्प के रूप में विकसित हुए हैं।

इक्विटी-लिंक्ड एन्युटी को वैरिएबल एन्युइटी(varible annuity) और इक्विटी-इंडेक्स एन्युइटी(equity indexed annuity) में विभाजित किया जा सकता है। इन दोनों के बीच की विविधता यह है कि इक्विटी-लिंक्ड एन्युटी पॉलिसीधारक को बाजार में निवेश का स्वामित्व रखने के लिए सक्षम बनाती है, जबकि वैरिएबल एन्युइटी में रिटर्न(return) अंतर्निहित सूचकांक जैसे S&P 500 आदि के प्रदर्शन के आधार पर पॉलिसीधारक के खाते में जमा किया जाता है। इसके अतिरिक्त, इक्विटी-इंडेक्स एन्युइटी(equity indexed annuity) मध्यम जोखिम वाले उत्पाद हैं, जबकि वैरिएबल एन्युइटी जोखिम में उच्च हैं।

इक्विटी-इंडेक्स एन्युइटी के कुछ लोकप्रिय डिजाइन में पॉइंट-टू-पॉइंट डिज़ाइन(point to point design) और वार्षिक रीसेट(annual reset) डिज़ाइन शामिल हैं।
पॉइंट-टू-पॉइंट डिज़ाइन में, दो तय पॉइंट के बीच की अंक वृद्धि के आधार पर रिटर्न जमा किया जाता है जबकि, वार्षिक रीसेट डिजाइन में, एक वर्ष का लाभ पॉलिसीधारकों को खाते में दिया जाता है और लॉक किया जाता हैं।

इसी तरह, वैरिएबल एन्युइटी एक जीवित या मृत्यु बेनिफिट राइडर(benefit rider) के साथ आती हैं, जहां जीवित बेनिफिट राइडर(living benefit rider) पॉलिसीधारक को जीवन भर की कमाई की गारंटी प्रदान करते हैं, और मृत्यु बेनिफिट राइडर(death benefit rider) पॉलिसीधारक को मृत्यु की कमाई की गारंटी देता है। जीवित बेनिफिट राइडर बाजार में भागीदारी के साथ पेंशन योजनाओं के समान है।

यह थीसिस(thesis) वैरिएबल एन्युइटी और इक्विटी-इंडेक्स एन्युइटी(equity indexed annuity) के मूल्य निर्धारण की समस्या को संबोधित करता है। यह थीसिस(thesis) इन एन्युइटी के मूल्य निर्धारण के लिए विभिन्न स्टोकेस्टिक मॉडल(stochastic model) प्रस्तावित करता है।

इक्विटी-लिंक्ड एन्युटी से जुड़ी मूल्य निर्धारण समस्या में तीन प्रकार के जोखिम शामिल हैं:

निवेश जोखिम(investment risk), ब्याज दर जोखिम (interest rate risk) और मृत्यु दर जोखिम(mortality risk)। इक्विटी रिटर्न में विभिन्न विशेषताएं शामिल हैं जैसे की वोलैटिलिटी क्लस्टरिंग(volatility clustering), लिवरेज इफ़ेक्ट(levrage effect), उच्च कुर्टोसिस(high kurtosis), आदि।

हम इन विशेषताओं को पकड़ने के लिए एक असममित गार्च मॉडल(asymmetric GARCH Model) प्रस्तावित करते हैं और एक मूल्य निर्धारण मॉडल को प्रस्तावित करते हैं। 2007-08 के वित्तीय संकट के दौरान, कई वैरिएबल एन्युइटी(variable annuity) उत्पाद लोकप्रिय और मूल्यवान हो गए। बाजार की कीमतों में गिरावट से कई ऐसे बीमा विक्रेता को नुकसान हुआ जो जीवित गारंटी को पूरा करने में असमर्थ थे। घाटे के कई कारणों में से एक है- उत्पादों का उचित मूल्य निर्धारण। इसलिए, यह थीसिस जीवित बेनिफिट राइडर के मूल्य निर्धारण के लिए एक जंप-डायजेशन(jump diffusion) मॉडल का प्रस्ताव करती है। इसके अलावा, हमने एक गोम्पर्टज़ मॉडल(Gompertz Model) द्वारा मृत्यु दर जोखिम का मॉडल तैयार किया, और हमने ब्याज दर का भी मॉडल बनाया।

इसके अलावा, हमने पॉलिसीधारक के लिए उपलब्ध अतिरिक्त लाभों में से कुछ को मॉडल किया है, जैसे कि आत्मसमर्पण विकल्प(surrender option) और रोल-अप विकल्प(roll up option)। रिटर्न में लंबी दूरी की निर्भरता(long-range dependency) को पकड़ने के लिए, मिक्स्ड फ्रॅक्शनल ब्रोनियन मोशन मॉडल (Mixed Fraction brownian motion model) प्रस्तावित किया है।

लोकप्रिय ली-कार्टर(Lee-carter) मॉडल की विविधताओं को मृत्यु दर(mortality rate) मॉडलिंग और इसलिए मूल्य निर्धारण(pricing) के लिए उपयोग किया गया है। साहित्य से, यह स्पष्ट है कि यह प्रारंभिक कूद(initial jumps) नहीं है जिसके परिणामस्वरूप बाजार में मंदी आती है, यह समय के साथ इन छलांगों का आयाम है। इसलिए इस थीसिस(thesis) के अंत में, इक्विटी-इंडेक्स एन्युइटी के मूल्यांकन के लिए एक हॉक्स जम्प -डिफ्फुसिओं मॉडल(hawkes jump diffusion model) प्रस्तावित किया है।

इन उत्पादों की बड़ी अवधि के कारण, हमने ब्याज दर जोखिम की मॉडलिंग को वासिसक मॉडल(Vasicek Model) का उपयोग करके प्रस्तावित किया है। हॉक्स जम्प -डिफ्फुसिओं मॉडल(hawkes jump diffusion model) जंप क्लस्टरिंग (jump clustering) कैप्चर कर सकता है और इसलिए इक्विटी-इंडेक्स एन्युइटी का उचित मूल्य प्रदान कर सकता है।

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(x)	life aged x
${}_tp_x$	the probability that (x) will survive at least t years
${}_tq_x$	the probability that (x) will die within t years given by $${}_tq_x = 1 -_t p_x$$
${}_{s\|t}q_x$	the probability that (x) will survive s years and subsequently die within t years $${}_{s\|t}q_x =_s p_x \; {}_tq_{x+s}$$
μ_x	force of mortality or instantaneous rate of mortality at age x
τ_x	future lifetime of (x)
K_x	number of future years completed by (x) prior to death $$K_x = \lfloor \tau_x \rfloor$$
r	risk-free rate
$F(t)$	cumulative distribution function of τ_x, given by $$F(t) =_t q_x$$
$f(t)$	probability density function of T_x, given by $$f(t) = {}_tp_x\mu_{x+t}$$
B_t	standard Brownian motion
μ	drift rate
σ	fund volatility
δ	insurance fees
W_0	initial fund value
W_t	fund value at time t
G	the withdrawal amount fixed to be $g\%$ of W_0

List of Abbreviations

AIC	Akaike Information Criterion
AR	Autoregressive
ARCH	Autoregressive conditional heteroscedastic
BIC	Bayesian Information Criterion
BM	Brownian motion
DB	Death benefit
E-GARCH	Exponential GARCH
EIA	Equity-indexed annuity
GARCH	Generalized ARCH
GBM	Geometric Brownian motion
GJR-GARCH	Glosten-Jagannathan-Runkle GARCH
GLWB	Guaranteed lifelong withdrawal benefit
GMAB	Guaranteed minimum accumulation benefit
GMDB	Guaranteed minimum death benefit
GMIB	Guaranteed minimum income benefit
GMLB	Guaranteed minimum living benefit
GMWB	Guaranteed minimum withdrawal benefit
JD	Jump diffusion
JDBM	Jump diffusion Brownian motion
JDMFBM	Jump diffusion mixed fractional Brownian motion
LB	Living benefits
LLK	Log-Likelihood
MA	Moving Average
MFBM	Mixed fractional Brownian motion
SIC	Schwartz Information Criterion
T-GARCH	Threshold GARCH
VA	Variable annuity