A CHARACTERIZATION OF NORMAL 3-PSEUDOMANIFOLDS WITH A FEW SINGULARITIES

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A Characterization of Normal 3-Pseudomanifolds With a Few Singularities

by

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Dedicated to

My Parents and My Sister

Certificate

This is to certify that the thesis entitled **"A Characterization of Normal 3-Pseudomanifolds With a Few Singularities"** submitted by **Mr. Raju Kumar Gupta** to the **Indian Institute of Technology Delhi**, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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Abstract

Characterizing face-number-related invariants of a given class of simplicial complexes has been a central topic in combinatorial topology. In this regard, one well-known invariant for a d-dimensional finite simplicial complex K is g_2, defined by $g_2 := f_1 - (d+1)f_0 + \binom{d+2}{2}$, where f_0 and f_1 denote the number of vertices and edges in K. A *normal d-pseudomanifold* without boundary (respectively with boundary) is a connected pure simplicial complex in which every face of dimension $(d-1)$ is contained in exactly two (respectively at most two) facets and the links of all the simplices of dimension $\leq (d-2)$ are connected. In a normal d-pseudomanifold K, the link of a vertex is a normal $d-1$-pseudomanifold. Vertices in K whose links are triangulated spheres are called *non-singular* vertices, and the remaining are called *singular* vertices.

Let K be a normal 3-pseudomanifold such that $g_2(K) \leq g_2(\mathrm{lk}\,(v)) + 9$ for some vertex v in K. Suppose that either K has only one singularity or K has two singularities (at least) one of which is an $\mathbb{RP}^2$-singularity. We prove that K is obtained from some boundary complexes of 4-simplices by a sequence of operations of types connected sums, bistellar 1-moves, edge contractions, edge expansions, vertex foldings, and edge foldings. In case K has one singularity, $|K|$ is a handlebody with its boundary coned off. Further, we prove that the above upper bound is sharp for such normal 3-pseudomanifolds. These results are in Chapter 3.

In Chapter 4, we characterize a normal 3-pseudomanifold K with $g_2(K) \leq 4$. We know that if a normal 3-pseudomanifold K with $g_2(K) \leq 4$ does not have any singular vertices, then it is a triangulated 3-sphere. First, we prove that a normal 3-pseudomanifold K with $g_2(K) \leq 4$ has at most two singular vertices. Next, we prove that a normal 3-pseudomanifold K with $g_2(K) \leq 4$, which is not a triangulated 3-sphere, is obtained from some boundary complexes of 4-simplices by a sequence of operations such as connected sums, edge expansions, and an edge folding. Furthermore, we prove that the normal 3-pseudomanifold K is a triangulation of the suspension of $\mathbb{RP}^2$. Additionally, using [73], we reframe the characterization of a normal 3-pseudomanifold K with no singular vertices for $g_2(K) \leq 9$.

Next, in Chapter 5, we extend the results obtained in Chapter 4. We characterize normal 3-pseudomanifolds with $g_2 = 5$. First, we show that a normal 3-pseudomanifold K with $g_2 = 5$ has no more than two singular vertices. Then, we show that K is obtained from some boundary complexes of 4-simplices by a sequence of possible operations of types connected sums, bistellar 1-moves, edge contractions, edge expansions, and an edge folding. As a result, K is a triangulation of either a sphere or a suspension of $\mathbb{RP}^2$.

The *average edge order*, denoted by $\mu_0(K)$, of a normal 3-pseudomanifold is defined as $\mu_0(K) = \frac{3f_2(K)}{f_1(K)}$, where $f_2(K)$ and $f_1(K)$ represent the number of faces and edges in K, respectively. In our study, we establish that, for a normal 3-pseudomanifold K with singularities, $\mu_0(K) \geq \frac{30}{7}$. Moreover, equality holds if and only if K is a one-vertex suspension of $\mathbb{RP}^2$ with seven vertices. Furthermore, we establish that when $\frac{30}{7} \leq \mu_0(K) \leq \frac{9}{2}$, K can be derived from some boundary complexes of 4-simplices by a sequence of possible operations, including connected sums, bistellar 1-moves, edge contractions, edge expansions, a vertex folding, and an edge folding. In the end, we discuss some normal 3-pseudomanifolds exhibiting higher average edge orders. These results are given in Chapter 6.

सार

किसी दी गई सिंप्लीशियल कॉम्प्लेक्स के फेस-संख्या से संबंधित गुणों का अध्ययन कॉम्बिनेटरियल टोपोलॉजी में एक महत्वपूर्ण विषय रहा है। इस संदर्भ में, एक d-आयामी सीमित सिंप्लीशियल कॉम्प्लेक्स K के लिए एक प्रमुख इनवेरिएंट g_2 है, जिसे निम्नलिखित रूप में परिभाषित किया गया है: $g_2 := f_1 - (d+1)f_0 + \binom{d+2}{2}$, जहाँ f_0 और f_1 क्रमशः K में शीर्षों और किनारों की संख्या को दर्शाते हैं।

एक सामान्य d-स्यूडोमैनिफोल्ड (सीमा सहित) एक संयोजित और शुद्ध सिंप्लीशियल कॉम्प्लेक्स होता है, जिसमें प्रत्येक $(d-1)$-आयामी फेस ठीक दो (या अधिकतम दो) फ़ेसेट्स में निहित होता है, और सभी $\leq (d-2)$-आयामी सिंप्लिसेस के लिंक जुड़े होते हैं। सामान्य d-स्यूडोमैनिफोल्ड K में किसी शीर्ष का लिंक सामान्य $(d-1)$-स्यूडोमैनिफोल्ड होता है। K में वे शीर्ष जिनके लिंक ट्रायंगुलेटेड गोले होते हैं, नॉन-सिंगुलर शीर्ष कहलाते हैं, जबकि शेष शीर्ष सिंगुलर कहलाते हैं।

हमने दिखाया है कि यदि K एक सामान्य 3-स्यूडोमैनिफोल्ड है और किसी शीर्ष v के लिए $g_2(K) \leq g_2((v)) + 9$ है, तो निम्नलिखित विशेषताएँ सत्य होती हैं: यदि K में केवल एक सिंगुलैरिटी है, तो $|K|$ एक हैंडलबॉडी है जिसकी सीमा कोंड ऑफ़ होती है। यदि K में दो सिंगुलैरिटीज़ हैं और उनमें से एक $\mathbb{RP}^2$-सिंगुलैरिटी है, तो K को कुछ 4-आयामी सिंप्लिसेस के बाउंड्री कॉम्पलेक्सेस पर कनेक्टेड समों, बाइस्टेलर 1-मूव्स, एज कॉन्ट्रैक्शंस, एज एक्सपेंशंस, शीर्ष फ़ोल्डिंगस, और एज फ़ोल्डिंगस जैसे ऑपरेशन्स की एक अनुक्रम द्वारा प्राप्त किया जा सकता है। उपरोक्त इनेकुअलिटी $g_2(K) \leq g_2((v)) + 9$ सामान्य 3-स्यूडोमैनिफोल्ड्स के लिए इष्टतम है। ये परिणाम अध्याय 3 में दिए गए हैं।

अध्याय 4 में, हम $g_2(K) \leq 4$ वाली एक सामान्य 3-स्यूडोमैनिफोल्ड K का वर्णन करते हैं। हम जानते हैं कि यदि $g_2(K) \leq 4$ वाली कोई सामान्य 3-स्यूडोमैनिफोल्ड K में कोई सिंगुलर शीर्ष नहीं है, तो वह एक ट्रायंगुलेटेड गोला होता है। सबसे पहले, हम सिद्ध करते हैं कि $g_2(K) \leq 4$ वाली एक सामान्य 3-स्यूडोमैनिफोल्ड K में अधिकतम दो सिंगुलर शीर्ष होते हैं। फिर, हम सिद्ध करते हैं कि $g_2(K) \leq 4$, जो एक ट्रायंगुलेटेड गोला नहीं है, उसे कुछ 4-आयामी सिंप्लिसेस के बाउंड्री कॉम्पलेक्सेस पर कनेक्टेड समों, एज एक्सपेंशंस, और एक एज फ़ोल्डिंग लगाकर प्राप्त किया जा सकता है। इसके अतिरिक्त, हम सिद्ध करते हैं कि सामान्य 3-स्यूडोमैनिफोल्ड $|K|$, $\mathbb{RP}^2$ का सस्पेंशन है। इसके अलावा, डी. वॉकप के परिणामों का उपयोग करके, हम $g_2(K) \leq 9$ के लिए सामान्य 3-स्यूडोमैनिफोल्ड K के बिना सिंगुलर शीर्षों के वर्णन को पुनः प्रस्तुत करते हैं।

अगले अध्याय 5 में, हम अध्याय 4 में प्राप्त परिणामों का विस्तार करते हैं। हम $g_2 = 5$ वाली सामान्य 3-स्यूडोमैनिफोल्ड्स का वर्णन करते हैं। सबसे पहले, हम यह दिखाते हैं कि $g_2 = 5$ वाली एक

सामान्य 3-स्यूडोमैनिफोल्ड K में अधिकतम दो सिंगुलर शीर्ष होते हैं। फिर, हम दिखाते हैं कि K को कुछ 4-आयामी सिंप्लिसेस के बाउंड्री कॉम्पलेक्सेस पर कनेक्टेड समों, बाइस्टेलर 1-मूव्स, एज कॉन्ट्रैक्शंस, एज एक्सपेंशंस, और एज फ़ोल्डिंगस जैसे ऑपरेशन्स लगाकर प्राप्त किया जा सकता है। परिणामस्वरूप, $|K|$ या तो एक गोलक या $\mathbb{RP}^2$ का सस्पेंशन होता है।

एवरेज एज ऑर्डर, जिसे $\mu_0(K)$ द्वारा निरूपित किया जाता है, एक सामान्य 3-स्यूडोमैनिफोल्ड का परिभाषित रूप है: $\mu_0(K) = \frac{3f_2(K)}{f_1(K)}$, जहाँ $f_2(K)$ और $f_1(K)$ क्रमशः K में फेस और किनारों की संख्या को दर्शाते हैं। हमारे अध्ययन में, हम स्थापित करते हैं कि, एक सामान्य 3-स्यूडोमैनिफोल्ड K के लिए जिसमें सिंगुलरिटीज़ होती हैं, उसमें $\mu_0(K) \geq \frac{30}{7}$ होता है। इसके अलावा, समानता तभी होती है जब और केवल तब, K ट्रायंगुलेटेड $\mathbb{RP}^2$ का एक एक-शीर्ष सस्पेंशन हो, जिसमें सात शीर्ष होते हैं। इसके अलावा, हम यह स्थापित करते हैं कि जब $\frac{30}{7} \leq \mu_0(K) \leq \frac{9}{2}$ है, तो K को कुछ 4-आयामी सिंप्लिसेस के बाउंड्री कॉम्पलेक्सेस पर कनेक्टेड समों, बाइस्टेलर 1-मूव्स, एज कॉन्ट्रैक्शंस, एज एक्सपेंशंस, शीर्ष फ़ोल्डिंगस, और एज फ़ोल्डिंगस जैसे ऑपरेशन्स की एक अनुक्रम द्वारा प्राप्त किया जा सकता है। अंत में, हम कुछ सामान्य 3-स्यूडोमैनिफोल्ड्स पर चर्चा करते हैं जो उच्च एवरेज एज ऑर्डर प्रदर्शित करते हैं। ये परिणाम अध्याय 6 में दिए गए हैं।

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Symbol	Meaning
$\mathbb{R}^n$	n-dimensional Euclidean Space,
$\mathbb{N}$	Set of natural numbers,
$\mathbb{Z}$	Set of integers,
$=$	equal to,
$\neq$	not equal to,
$\exists$	there exists,
$\nexists$	there does not exists,
$\forall$	for all,
$\cup$	union,
$\cap$	intersection,
$\in$	belongs to,
$\notin$	does not belong to,
$\emptyset$	empty set,
$\cong$	isomrophic to,
$\#$	connected sum,
$A \subseteq X$	A is a subset of X,
$A \nsubseteq X$	A is not a subset of X,
$A \times B$	Cartesian product of the sets A and B,
$A \setminus B$	set containing elements of A but not B,
$\lceil \cdot \rceil$	the smallest integer greater than or equal to $(\cdot)$,
$\lfloor \cdot \rfloor$	the largest integer less than or equal to $(\cdot)$,
f_i	i-th coordinate of the f-vector,
g_i	i-th coordinate of the g-vector,

h_i	i-th coordinate of the h-vector,
β_i	i-th betti number,
$V(K)$	vertex set of a simplicial complex K,
$E(K)$	edge set of a simplicial complex K,
$C(K)$	cone over a simplicial complex K,
$\chi(K)$	Euler characteristic of a simplicial complex K,
$K_1 \star K_2$	join of simplicial complexes K_1 and K_2,
$\mathrm{lk}\,(\sigma, K)$	link of a simplex σ in a simplicial complex K,
$\mathrm{st}\,(\sigma, K)$	star of a simplex σ in a simplicial complex K,
$\partial(K)$	boundary of a simplicial complex K,
$\mathbb{S}^n$	n-sphere,
$\mathbb{S}^n_{n+2}$	triangulation of an n-sphere with n+2 vertices,
$\mathbb{RP}^n$	n-dimensional projective space,
$\mathbb{T}^2$	torus,
$d(\sigma, K)$ (or, $d(\sigma)$)	number of vertices in $\mathrm{lk}\,(\sigma)$,
$(x, y]$	semi-open and semi-closed edge xy, where $y \in (x, y]$ but $x \notin (x, y]$,
(x, y)	open edge xy, where $x, y \notin (x, y)$,
$B_{x_1,\ldots,x_m}(p; q)$	bi-pyramid with m base vertices $x_1, \ldots, x_m$ and apexes p and q.