

**GROUP THEORETIC APPROACHES FOR THE
SOLUTIONS OF NONLINEAR PARTIAL DIFFERENTIAL
EQUATIONS GOVERNING CERTAIN PHYSICAL SYSTEMS**

by

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Department of Mathematics

Submitted

in fulfilment of the requirements of the degree of

DOCTOR OF PHILOSOPHY

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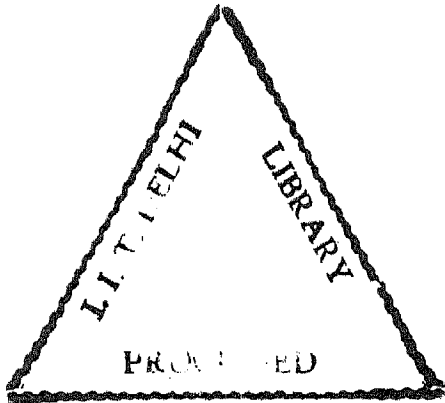
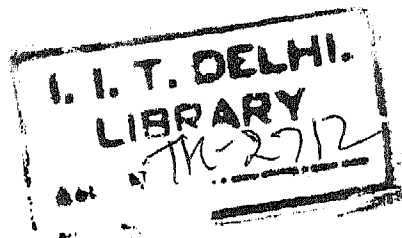


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Certificate

This is to certify that the thesis entitled *Group Theoretic Approaches for the Solutions of Nonlinear Partial Differential Equations Governing Certain Physical Systems* which is being submitted by *Karanjeet Singh* for the award of the degree of *Doctor of Philosophy in Mathematics* to the *Indian Institute of Technology, Delhi*, is a bonafide record of research work done under our guidance and supervision.

The thesis has reached the standard fulfilling the requirements of the regulations relating to the degree. The results obtained in the thesis have not been submitted to any other University or Institute for the award of any degree or diploma.



(O.P. BHUTANI)

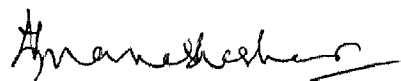
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*To my father whose memories give special
meaning to my work and life*

*To my wife and son for their love,
patience and sacrifice*

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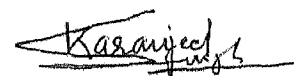
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KARANJEET SINGH

Abstract

The thesis entitled GROUP THEORETIC APPROACHES FOR THE SOLUTIONS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS GOVERNING CERTAIN PHYSICAL SYSTEMS comprises eight chapters. Chapter 1 presents primarily the review of the related works and the methodologies utilized in the thesis. The investigations carried out are confined to the applications of group-theoretic methods to two different categories of problems viz. systems governed by single nonlinear partial differential equation (pde) and those governed by a system of nonlinear partial differential equations (pdes), the thesis, accordingly, is divided into two parts, namely Part - I and Part - II. Part - I, consisting of Chapters 2-6, deals with single pde. Part - II, that includes Chapters 7-8, deals with the systems of pdes. Chapters 3 and 6 are based on the applications of method of isovectors while the remaining chapters utilize Steinberg's Symmetry Approach.

In Chapter 2, the invariance under continuous groups of transformations of a generalized K-dV-Burger type equation, involving an arbitrary function of spatial variable, has been studied. Particular cases corresponding to certain specific values of the parameters involved and those spatial forms for which the equation can be reduced to an ordinary differential equation (ode) are presented. Besides recovering certain available results, some interesting exact solutions are derived. We have, using the isovector method, re-examined in Chapter 3, the problem undertaken in Chapter 2. After constructing the components of the isovector field, the invariant groups of transformations are determined by means of orbital equations. These are further utilized to derive certain new exact solutions for different values of the parameters.

Chapter 4 is devoted to the study of quintic order variable coefficient nonlinear Schrödinger (VCNLS) equations. The coefficients are all assumed, in general, to be complex functions of spatial and temporal variables. Using the Symmetry Approach the most general system of determining equations for the continuous groups of transformations leaving the VCNLS equations invariant is obtained. The efforts are then concentrated on some specific models of physical interest. Some exact solutions and various travelling wave (TW) solutions for different models are reported.

The perfect fluid distributions expressed by a five-dimensional flat metric are either of Petrov-type O or of Petrov-type D. Almost all the solutions of type-O are known, but type-D solutions are yet to be exhausted. In Chapter 5, we study a problem recently

investigated. More specifically, the pde corresponding to the metric function in a five-dimensional flat space describing the perfect fluid distribution has been examined for generalized symmetries via Symmetry Approach. It is shown that the approach utilized here, besides providing generalized symmetries, for some particular values of the parameters involved, yields new metrics and Petrov-type D solutions.

In Chapter 6, we have investigated a nonlinear heat equation using the method of isovectors. Some important symmetry reductions and exact solutions for different choices of the source term are reported.

Often, the efforts to derive exact solutions to Einstein's exterior field equations are either through the conventional form of these equations, which is a coupled system of two pdes or through the Ernst's form, which is a single pde. In Chapter 7, a nonconventional form of Einstein's equations has been investigated using Symmetry Approach. The interesting outcome of the study is that it not only leads to the recovery of solutions obtained through the conventional form but also yields physically more relevant asymptotically flat solutions.

Chapter 8 is concerned with the study of Einstein - Maxwell field equations which are four coupled nonlinear pdes of second order in four unknowns. To our knowledge, no exact solution of this system has been found till date, though some approximate results via perturbation technique do exist. On carrying over the Symmetry Approach to this system we have derived the group of transformations admitted by the system under consideration. Further, the resulting system has been reduced to a system of two ordinary differential equations. In view of the non-triviality of the original problem, this system of ordinary differential equations greatly simplifies the task and may be of immense help for any further study by the researchers.

Equations are numbered as (Ch.S.E) where Ch represents the number of the Chapter, S the number of the section of Chapter Ch and E stands for the serial number of equation in the section S. The details of the calculations, mostly related to Fréchet derivative(s), are facilitated in the form of Appendix - Ch A, where Ch stands for the number of the respective chapter. Equations occurring in the body of the appendix are numbered as (Ch A.N), where Ch A stands for the appendix of chapter Ch and N denotes the number of the respective equation. The bibliography is provided at the end of the thesis. The number(s) in the bracket [] refers to the article(s)/book(s) mentioned in the bibliography.

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