GROUP THEORETIC APPROACHES FOR THE SOLUTIONS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS GOVERNING CERTAIN PHYSICAL SYSTEMS

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Certificate

This is to certify that the thesis entitled *Group Theoretic Approaches for the Solutions of Nonlinear Partial Differential Equations Governing Certain Physical Systems* which is being submitted by *Karanjeet Singh* for the award of the degree of *Doctor of Philosophy* in *Mathematics* to the *Indian Institute of Technology, Delhi*, is a bonafide record of research work done under our guidance and supervision.

The thesis has reached the standard fulfilling the requirements of the regulations relating to the degree. The results obtained in the thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

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To my father whose memories give special
meaning to my work and life

To my wife and son for their love,
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Abstract

The thesis entitled GROUP THEORETIC APPROACHES FOR THE SOLUTIONS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS GOVERNING CERTAIN PHYSICAL SYSTEMS comprises eight chapters. Chapter 1 presents primarily the review of the related works and the methodologies utilized in the thesis. The investigations carried out are confined to the applications of group-theoretic methods to two different categories of problems viz. systems governed by single nonlinear partial differential equation (pde) and those governed by a system of nonlinear partial differential equations (pdes), the thesis, accordingly, is divided into two parts, namely Part - I and Part - II. Part - I, consisting of Chapters 2-6, deals with single pde. Part - II, that includes Chapters 7-8, deals with the systems of pdes. Chapters 3 and 6 are based on the applications of method of isovectors while the remaining chapters utilize Steinberg's Symmetry Approach.

In Chapter 2, the invariance under continuous groups of transformations of a generalized K-dV-Burger type equation, involving an arbitrary function of spatial variable, has been studied. Particular cases corresponding to certain specific values of the parameters involved and those spatial forms for which the equation can be reduced to an ordinary differential equation (ode) are presented. Besides recovering certain available results, some interesting exact solutions are derived. We have, using the isovector method, re-examined in Chapter 3, the problem undertaken in Chapter 2. After constructing the components of the isovector field, the invariant groups of transformations are determined by means of orbital equations. These are further utilized to derive certain new exact solutions for different values of the parameters.

Chapter 4 is devoted to the study of quintic order variable coefficient nonlinear Schrödinger (VCNLS) equations. The coefficients are all assumed, in general, to be complex functions of spatial and temporal variables. Using the Symmetry Approach the most general system of determining equations for the continuous groups of transformations leaving the VCNLS equations invariant is obtained. The efforts are then concentrated on some specific models of physical interest. Some exact solutions and various travelling wave (TW) solutions for different models are reported.

The perfect fluid distributions expressed by a five-dimensional flat metric are either of Petrov-type O or of Petrov-type D. Almost all the solutions of type-O are known, but type-D solutions are yet to be exhausted. In Chapter 5, we study a problem recently

investigated. More specifically, the pde corresponding to the metric function in a five-dimensional flat space describing the perfect fluid distribution has been examined for generalized symmetries via Symmetry Approach. It is shown that the approach utilized here, besides providing generalized symmetries, for some particulars values of the parameters involved, yields new metrics and Petrov-type D solutions.

In Chapter 6, we have investigated a nonlinear heat equation using the method of isovectors. Some important symmetry reductions and exact solutions for different choices of the source term are reported.

Often, the efforts to derive exact solutions to Einstein's exterior field equations are either through the conventional form of these equations, which is a coupled system of two pdes or through the Ernst's form, which is a single pde. In Chapter 7, a nonconventional form of Einstein's equations has been investigated using Symmetry Approach. The interesting outcome of the study is that it not only leads to the recovery of solutions obtained through the conventional form but also yields physically more relevant asymptotically flat solutions.

Chapter 8 is concerned with the study of Einstein - Maxwell field equations which are four coupled nonlinear pdes of second order in four unknowns. To our knowledge, no exact solution of this system has been found till date, though some approximate results via perturbation technique do exist. On carrying over the Symmetry Approach to this system we have derived the group of transformations admitted by the system under consideration. Further, the resulting system has been reduced to a system of two ordinary differential equations. In view of the non-triviality of the original problem, this system of ordinary differential equations greatly simplifies the task and may be of immense help for any further study by the researchers.

Equations are numbered as (Ch.S.E) where Ch represents the number of the Chapter, S the number of the section of Chapter Ch and E stands for the serial number of equation in the section S. The details of the calculations, mostly related to Fréchet derivative(s), are facilitated in the form of Appendix - Ch A, where Ch stands for the number of the respective chapter. Equations occurring in the body of the appendix are numbered as (Ch A.N), where Ch A stands for the appendix of chapter Ch and N denotes the number of the respective equation. The bibliography is provided at the end of the thesis. The number(s) in the bracket [ ] refers to the article(s)/book(s) mentioned in the bibliography.

Contents

Chapter 1 *Introduction* *4*

1.1 Literature Survey and Motivation 4

1.2 Symmetry Approach 9

1.3 Isovector Method 13

1.4 Problems to be Considered 24

PART - I

Chapter 2 *On Certain Exact Solutions of a Generalized K-dV-Burger Type Equation via Symmetry Approach* *28*

2.1 Introduction 28

2.2 Determination of Symmetries 30

2.3 The Porous Medium Equation 32

2.4 The Dispersive Equation 44

2.5 The General Equation 49

2.6 Concluding Remarks 53

Appendix - 2A *55*

Chapter 3 *On Certain Exact Solutions of a Generalized K-dV-Burger Type Equation via Isovector Method* *59*

3.1 Introduction 59

3.2 Representation in Exterior Forms 60

3.3 Closed Ideal and Isovector Field 61

3.4 Similarity Reductions and Exact Solutions 68

3.5 Concluding Remarks 76

Chapter 4 *Symmetry Reductions and Invariant Solutions of Some Nonlinear Schrödinger Equations* *77*

4.1 Introduction 77

4.2 Determining Equations for Group Infinitesimals 79

4.3 Quintic order NLS equation 83
4.4 Cubic NLS equation 100
Appendix - 4A *110*

Chapter 5 *Generalized Similarity Solutions for the Type D Fluid in Five Dimensional Flat Space* *122*

5.1 Introduction 122
5.2 The Main Equations 123
5.3 Determination of Symmetries 124
5.4 Some Special Cases 127
5.5 Reduction to First Order Ordinary Differential Equation 132
5.6 Computing Flow Vector 133
5.7 Concluding Remarks 137
Appendix - 5A *138*

Chapter 6 *On Generalized Symmetries and Exact Solutions of $u_t = u_{xx} + f(u)$* *140*

6.1 Introduction 140
6.2 Exterior Differential Forms and Closed Ideal 142
6.3 Determination of Isovector Field 143
6.4 Special Cases 148

PART - II

Chapter 7 *Exact Solutions of Einstein Equations for Rotating Fields in Nonconventional Form* *161*

7.1 Introduction 161
7.2 Einstein Equations 162
7.3 Determination of Symmetries 165
7.4 Similarity Reduction and Exact Solutions 167
7.5 Concluding Remarks 172
Appendix - 7A *174*

Chapter 8 ***Symmetry Reductions of Einstein-Maxwell Field Equations*** ***179***

8.1 Introduction 179

8.2 Einstein-Maxwell Equations 180

8.3 Determination of Symmetries 182

8.4 Similarity Solutions and Similarity Reductions 187

Appendix - 8A *192*

Bibliography ***207***