MATCHING AND ITS VARIATIONS: AN ALGORITHMIC STUDY

JUHI CHAUDHARY

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY DELHI

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by

JUHI CHAUDHARY
Department of Mathematics

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Dedicated to

My Family

Certificate

This is to certify that the thesis entitled **"Matching and Its Variations: An Algorithmic Study"** submitted by **"Ms. Juhi Chaudhary"** to the **Indian Institute of Technology Delhi**, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bona fide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi
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Prof. Bhawani Sankar Panda
Professor
Department of Mathematics
IIT Delhi

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"To raise new questions, new possibilities, to regard old problems from a new angle, requires creative imagination and marks real advance in science"- Albert Einstein.

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Abstract

Matching is one of the oldest and widely studied areas of research in graph theory. A set $M \subseteq E$ of edges of a graph $G = (V, E)$ is called a *matching* if no two edges of M share a common vertex. Given a graph G, the Maximum Matching problem is the problem of finding a matching of maximum cardinality in G. The *matching number* of G is the maximum cardinality among all matchings in G and is denoted by $\mu(G)$. The edges in a matching M are called *matched* edges, and the remaining edges are called *unmatched* or $free$ edges. Given a matching M, a vertex $v \in V$ is called *M-saturated* if v is incident on an edge of M, that is, v is an end vertex of some edge of M. Given a graph $G = (V, E)$ and a matching M, we use the notation $V(M)$ to denote the M-saturated vertices and $G[V(M)]$ to denote the subgraph induced by the M-saturated vertices of G. A matching that saturates all the vertices of a graph is called a *perfect matching*.

In this thesis, we study the algorithmic aspects of some variations of matching, namely (i) Induced Matching, (ii) Dominating Induced Matching, (iii) Minimum Maximal Induced Matching, (iv) Acyclic Matching, (v) Minimum Maximal Acyclic Matching, and (vi) Minimum Maximal Uniquely Restricted Matching.

Let $M \subseteq E$ of the graph $G = (V, E)$. M is called an *induced matching* if, $G[V(M)]$, the subgraph induced by the M-saturated vertices of G is same as $G[M]$, where $G[M]$ is the subgraph of G induced by M. The Induced Matching problem asks to find an induced matching of maximum cardinality in G. The decision version of the INDUCED MATCHING problem is known to be NP-complete even for bipartite graphs. In this thesis, we strengthen the hardness result of the INDUCED MATCHING problem by proving it to be NP-complete for star-convex bipartite graphs, comb-convex bipartite graphs, and perfect elimination bipartite graphs. On the other hand, we study the weighted version of the INDUCED MATCHING problem in graph classes, namely circular-convex bipartite graphs and triad-convex bipartite graphs.

An edge e is said to be dominated by a matching M if it either belongs to M or is adjacent to an edge of M. M is called a *dominating induced matching* if every edge of G is dominated by the edges of M exactly once. The *dominating induced matching* (DIM) problem asks to compute a dominating induced matching (*d.i.m.*) in a graph G that admits a dominating induced matching. The decision version of the DIM problem is known to be NP-complete even for bipartite graphs. In this thesis, we study the DIM problem in various subclasses of bipartite graphs, namely perfect elimination bipartite graphs, star-convex bipartite graphs, long-k-star-convex bipartite graphs, and circular-convex bipartite graphs.

M is called a *maximal induced matching* if M is an induced matching and it is not contained in any other induced matching of G. The Minimum Maximal Induced Matching (MIN-MAX-IND MATCHING) problem asks to find a maximal induced matching M of minimum cardinality in G. The decision version of the MIN-MAX-IND MATCHING problem is known to be NP-complete even for bipartite graphs. In this thesis, we strengthen the hardness result of the MIN-MAX-IND MATCHING problem by proving it to be NP-complete for perfect elimination bipartite graphs, star-convex bipartite graphs, and dually chordal graphs. We also establish the hardness difference between the INDUCED MATCHING problem and the MIN-MAX-IND MATCHING problem.

M is called an *acyclic matching* if, $G[V(M)]$, the subgraph induced by the M-saturated vertices of G is acyclic. The ACYCLIC MATCHING problem asks to find an

acyclic matching of maximum cardinality in G. The decision version of the ACYCLIC MATCHING problem is known to be NP-complete even for bipartite graphs. In this thesis, we study the ACYCLIC MATCHING problem in various subclasses of chordal graphs like split graphs, proper interval graphs, and block graphs. We also study the ACYCLIC MATCHING problem in various subclasses of bipartite graphs like bipartite permutation graphs and star-convex bipartite graphs. We also obtain some hardness and approximation results of the ACYCLIC MATCHING problem.

M is called a *maximal acyclic matching* if M is an acyclic matching and it is not properly contained in any other acyclic matching of G. The Minimum Maximal Acyclic Matching (MIN-MAX-ACY MATCHING) problem asks to find a maximal acyclic matching M of minimum cardinality in G. The decision version of the MIN-MAX-ACY MATCHING problem is known to be NP-complete even for bipartite graphs. In this thesis, we study the MIN-MAX-ACY MATCHING problem in graph classes, namely bipartite graphs, perfect elimination bipartite graphs, and proper interval graphs. We also establish the hardness difference between the ACYCLIC MATCHING problem and the MIN-MAX-ACY MATCHING problem. We also obtain some hardness and approximation results of the MIN-MAX-ACY MATCHING problem.

M is called a *uniquely restricted matching* if, $G[V(M)]$, the subgraph induced by the M-saturated vertices of G contains exactly one perfect matching. M is called a *maximal uniquely restricted matching* if M is a uniquely restricted matching and it is not properly contained in any other uniquely restricted matching of G. The Minimum Maximal Uniquely Restricted Matching (MIN-MAX-UR MATCHING) problem asks to find a maximal uniquely restricted matching M of minimum cardinality in G. The decision version of the MIN-MAX-UR MATCHING problem is known to be NP-complete even for bipartite graphs. In this thesis, we study the MIN-MAX-UR MATCHING problem in various subclasses of bipartite graphs and chordal graphs.

We propose some linear-time algorithms in graph classes, namely bipartite permutation graphs, proper interval graphs, and threshold graphs. We also obtain some hardness and approximation results of the MIN-MAX-UR MATCHING problem.

Most of the computational graph-theoretic problems are NP-hard for general graphs, and the graphs used in models possess some special structures. Therefore, the study of NP-hard graph problems restricted to special graph classes has generated a lot of interest. In this thesis, we strengthen the existing literature of the above-mentioned variations of matching by studying the complexity status of the problems associated with them in various important classes of graphs. We propose some polynomial-time algorithms to solve these problems for certain graph classes. We strengthen the hardness result of the problems mentioned. We also propose some approximation algorithms and establish some approximation hardness results for the problems mentioned above.

सार

ग्राफ सिद्धांत में अनुसंधान के सबसे पुराने और व्यापक रूप से अध्ययन किए गए क्षेत्रों में से एक **मिलान** है। एक ग्राफ G को देखते हुए, **अधिकतम मिलान समस्या** है: G में अधिकतम कार्डिनैलिटी का मिलान खोज करने की समस्या। मिलान संख्या G, G में सभी मिलानों में अधिकतम कार्डिनैलिटी है और इसे $\mu(G)$ द्वारा निरूपित किया जाता है। एक मिलान M के किनारों को **मिला हुआ** किनारा कहा जाता है, और शेष किनारों को **बेजोड़** या **मुक्त** किनारा कहा जाता है । एक मिलान M को देखते हुए, एक शीर्ष $v \in V$ को ***M*- संतृप्त** कहा जाता है यदि v, M के किनारे पर आपतित है, अर्थात v, M के किसी किनारे का अंतिम शीर्ष है। एक ग्राफ $G = (V, E)$ और एक मिलान M के लिए हम ***M*- संतृप्त** शीर्षों के समूह को दर्शाने के लिए अंकन $V(M)$ का उपयोग करते हैं . $V(M)$ द्वारा प्रेरित उपग्राफ को दर्शाने के लिए अंकन $G[V(M)]$ का उपयोग करते हैं । एक मिलान जो एक ग्राफ के सभी शीर्षों को संतृप्त करता है, एक **पूर्ण मिलान** कहलाता है ।

इस थीसिस में, हम मिलान के कुछ रूपों के एल्गोरिथम पहलुओं का अध्ययन करते हैं, अर्थात् (i) प्रेरित मिलान, (ii) हावी प्रेरित मिलान, (iii) न्यूनतम अधिकतम प्रेरित मिलान, (iv) एसाइक्लिक मिलान, (v) न्यूनतम अधिकतम एसाइक्लिक मिलान , और (vi) न्यूनतम अधिकतम विशिष्ट रूप से प्रतिबंधित मिलान।

ग्राफ $G = (V, E)$ के उपसमुच्चय $M \subseteq E$ को **प्रेरित मिलान** कहा जाता है यदि, $G[V(M)]$, G के M -संतृप्त शीर्षों द्वारा प्रेरित उपग्राफ $G[M]$ के समान है, जहां $G[M]$, M द्वारा प्रेरित G का उप-ग्राफ है। **प्रेरित मिलान समस्या**, G में अधिकतम कार्डिनैलिटी का एक प्रेरित मिलान खोजने के लिए कहती है। प्रेरित मिलान समस्या का निर्णय संस्करण एनपी-पूर्ण होने के लिए जाना जाता है, यहां तक कि द्विदलीय ग्राफ के लिए भी। इस थीसिस में, हम प्रेरित मिलान समस्या के कठोरता परिणाम को स्टार-उत्तल द्विदलीय ग्राफ़, कंघी-उत्तल द्विदलीय ग्राफ़ और पूर्ण उन्मूलन द्विदलीय ग्राफ़ के लिए एनपी-पूर्ण साबित करके मजबूत करते हैं। दूसरी ओर, हम गोलाकार-उत्तल द्विदलीय ग्राफ़ और त्रय-उत्तल द्विदलीय ग्राफ़ कक्षाओं में प्रेरित मिलान समस्या का भारित संस्करण अध्ययन करते हैं।

एक किनारे e को मेल खाने वाले M का प्रभुत्व कहा जाता है यदि यह या तो M के भीतर है या M के किनारे के निकट है। M को **हावी प्रेरित मिलान** कहा जाता है यदि G के प्रत्येक किनारे पर M के किनारों पर एक बार प्रभुत्व होता है। **हावी प्रेरित मिलान (डीआईएम) समस्या** एक हावी प्रेरित मिलान (डीआईएम) की गणना उन ग्राफ G में करने के लिए कहती है जो एक हावी प्रेरित मिलान को स्वीकार करता है। डीआईएम समस्या का निर्णय संस्करण द्विदलीय ग्राफ़ के लिए भी एनपी-पूर्ण होने के लिए जाना जाता है। इस थीसिस में, हम द्विदलीय ग्राफ के विभिन्न उपवर्गों में डीआईएम

समस्या का अध्ययन करते हैं, अर्थात् पूर्ण उन्मूलन द्विदलीय ग्राफ, स्टार-उत्तल द्विदलीय ग्राफ, लंबे-के-स्टार-उत्तल द्विदलीय ग्राफ और गोलाकार -उत्तल द्विदलीय ग्राफ।

M को **अधिकतम प्रेरित मिलान** कहा जाता है यदि M एक प्रेरित मिलान है और यह G के किसी अन्य प्रेरित मिलान में पूर्णता से समाहित नहीं है। **न्यूनतम अधिकतम प्रेरित मिलान (न्यूनतम-अधिकतम-इंड मिलान) समस्या** अधिकतम प्रेरित मिलान M की कार्डिनैलिटी को न्यूनतम करने के लिए कहती है। न्यूनतम-अधिकतम-इंड मिलान समस्या का निर्णय संस्करण एनपी-पूर्ण होने के लिए जाना जाता है, यहां तक कि द्विदलीय ग्राफ के लिए भी। इस थीसिस में, हम न्यूनतम-अधिकतम-इंड मिलान समस्या के कठोरता परिणाम को पूर्ण उन्मूलन द्विदलीय ग्राफ, स्टार-उत्तल द्विदलीय ग्राफ और दोहरी कॉर्डल ग्राफ के लिए एनपी-पूर्ण साबित करके मजबूत करते हैं। हम प्रेरित मिलान समस्या और न्यूनतम-अधिकतम-इंड मिलान समस्या के बीच की कठोरता अंतर भी स्थापित करते हैं ।

M को **एसाइक्लिक मिलान** कहा जाता है यदि, $G[V(M)]$, G के M- संतृप्त शीर्षों द्वारा प्रेरित उप-लेख एसाइक्लिक है। **एसाइक्लिक मिलान समस्या,** G में अधिकतम कार्डिनैलिटी के एसाइक्लिक मिलान के लिए पूछती है। एसाइक्लिक मिलान समस्या का निर्णय संस्करण एनपी-पूर्ण होने के लिए जाना जाता है, यहां तक कि द्विदलीय ग्राफ के लिए भी। इस थीसिस में, हम कॉर्डल ग्राफ के विभिन्न उपवर्गों जैसे स्प्लिट ग्राफ, उचित अंतराल ग्राफ और ब्लॉक ग्राफ में एसाइक्लिक मिलान समस्या का अध्ययन करते हैं। हम द्विदलीय ग्राफ़ के विभिन्न उपवर्गों जैसे द्विदलीय क्रमपरिवर्तन ग्राफ़ और स्टार-उत्तल द्विदलीय ग्राफ़ में एसाइक्लिक मिलान समस्या का भी अध्ययन करते हैं। हम एसाइक्लिक मिलान समस्या के कुछ कठोरता और सन्निकटन परिणाम भी प्राप्त करते हैं।

M को **न्यूनतम अधिकतम एसाइक्लिक मिलान** कहा जाता है यदि M एक एसाइक्लिक मिलान है और यह G के किसी अन्य एसाइक्लिक मिलान में पूर्णता से समाहित नहीं है। **न्यूनतम अधिकतम एसाइक्लिक मिलान (न्यूनतम-अधिकतम-ऐसी मिलान) समस्या** अधिकतम एसाइक्लिक मिलान M की कार्डिनैलिटी को न्यूनतम करने के लिए कहती है।। न्यूनतम-अधिकतम-एसी मिलान समस्या का निर्णय संस्करण एनपी-पूर्ण होने के लिए जाना जाता है, यहां तक कि द्विदलीय ग्राफ के लिए भी। इस थीसिस में, हम ग्राफ कक्षाओं में न्यूनतम-अधिकतम-एसी मिलान समस्या का अध्ययन करते हैं, अर्थात् द्विदलीय ग्राफ, पूर्ण उन्मूलन द्विदलीय ग्राफ और उचित अंतराल ग्राफ। हम एसाइक्लिक मिलान समस्या और न्यूनतम-अधिकतम-ऐसी मिलान समस्या के बीच कठोरता अंतर भी स्थापित करते हैं। हम न्यूनतम-अधिकतम-ऐसी मिलान समस्या के कुछ कठोरता और सन्निकटन परिणाम भी प्राप्त करते हैं।

M को **विशिष्ट रूप से प्रतिबंधित मिलान** कहा जाता है, यदि $G[V(M)]$, G के M-संतृप्त शीर्षों द्वारा प्रेरित उप-ग्राफ में ठीक एक पूर्ण मिलान होता है। M को अधिकतम विशिष्ट रूप से प्रतिबंधित मिलान कहा जाता है यदि M एक विशिष्ट रूप से प्रतिबंधित मिलान है और यह G के किसी अन्य विशिष्ट रूप से प्रतिबंधित मिलान में पूर्णता से समाहित नहीं है। **न्यूनतम अधिकतम विशिष्ट रूप से प्रतिबंधित मिलान (न्यूनतम-अधिकतम-यूआर मिलान) समस्या** अधिकतम विशिष्ट रूप से प्रतिबंधित मिलान M की कार्डिनैलिटी को न्यूनतम करने के लिए कहती है । न्यूनतम-अधिकतम-यूआर मिलान समस्या का निर्णय संस्करण एनपी-पूर्ण होने के लिए जाना जाता है, यहां तक कि द्विपक्षीय ग्राफ के लिए भी। इस थीसिस में, हम न्यूनतम-अधिकतम-यूआर मिलान समस्या का अध्ययन करते हैं द्विदलीय ग्राफ और कॉर्डल ग्राफ के विभिन्न उपवर्गों में। हम ग्राफ कक्षाओं में कुछ रैखिक समय एल्गोरिदम का प्रस्ताव करते हैं, अर्थात् द्विदलीय क्रमपरिवर्तन ग्राफ़, उचित अंतराल ग्राफ़ और थ्रेशोल्ड ग्राफ़। हम न्यूनतम-अधिकतम-यूआर मिलान समस्या के कुछ कठोरता और सन्निकटन परिणाम भी प्राप्त करते हैं।

अधिकांश कम्प्यूटेशनल ग्राफ-सैद्धांतिक समस्याएं सामान्य ग्राफ़ के लिए एनपी-हार्ड हैं, और मॉडल में उपयोग किए जाने वाले ग्राफ़ में कुछ विशेष संरचनाएं होती हैं। इसलिए, विशेष ग्राफ वर्गों तक सीमित एनपी-हार्ड ग्राफ समस्याओं के अध्ययन ने बहुत रुचि पैदा की है। इस थीसिस में, हम विभिन्न महत्वपूर्ण वर्गों के रेखांकन में उनसे जुड़ी समस्याओं की जटिलता की स्थिति का अध्ययन करके मिलान की उपर्युक्त विविधताओं के मौजूदा साहित्य को मजबूत करते हैं। हम कुछ ग्राफ वर्गों के लिए इन समस्याओं को हल करने के लिए कुछ बहुपद-समय एल्गोरिदम प्रस्तावित करते हैं। हम उल्लिखित समस्याओं की कठोरता को मजबूत करते हैं। हम कुछ सन्निकटन एल्गोरिदम भी प्रस्तावित करते हैं और ऊपर वर्णित समस्याओं के लिए कुछ सन्निकटन कठोरता परिणाम स्थापित करते हैं।

Contents

Certificate **i**

Acknowledgements **iii**

Abstract **v**

List of Figures **xv**

List of Symbols **xix**

1 Introduction **1**

1.1 Introduction 1

1.2 Variations of Matching Studied in the Thesis 4

1.2.1 Induced Matching 5

1.2.2 Dominating Induced Matching 7

1.2.3 Minimum Maximal Induced Matching 9

1.2.4 Acyclic Matching 10

1.2.5 Minimum Maximal Acyclic Matching 11

1.2.6 Minimum Maximal Uniquely Restricted Matching 13

1.3 Preliminaries 15

1.3.1 Graph-theoretic Definitions and Notations 15

1.3.2 Algorithmic Preliminaries 17

1.4 Some Decision and Optimization Problems to be Used in the Thesis . 20

1.5 Graph Classes Studied in the Thesis 21

1.5.1 Bipartite graphs 21

1.5.2 Chordal graphs 27

1.5.3 Dually Chordal graphs 31

1.5.4 Planar graphs 31

1.5.5 Complement-reducible graphs 32

1.6 Contributions of the Thesis 32

1.7 Organization of the Thesis 35

2 Induced Matching 39

2.1 Introduction 39

2.2 NP-completeness Results 40

2.2.1 NP-completeness Proof for Star-convex Bipartite Graphs . . . 40

2.2.2 NP-completeness Proof for Comb-convex Bipartite Graphs . . 42

2.2.3 NP-completeness Proof for Perfect Elimination Bipartite Graphs 43

2.3 Induced Matching in Circular-convex Bipartite Graphs 45

2.4 Induced Matching in Triad-convex Bipartite Graphs 48

3 Dominating Induced Matching 59

3.1 Introduction 59

3.2 Some Observations for Dominating Induced Matching 60

3.3 NP-completeness Result 62

3.4 Dominating Induced Matching in Star-convex Bipartite Graphs . . . 64

3.4.1 Characterization-I of star-convex bipartite graphs admitting a *d.i.m.* 64

3.4.2 Characterization-II of star-convex bipartite graphs admitting a *d.i.m.* 75

3.5 Dominating Induced Matching in Long-k-star-convex Bipartite Graphs 81

3.6 Dominating Induced Matching in Circular-convex Bipartite Graphs . 87

4 Minimum Maximal Induced Matching 93

4.1 Introduction 93

4.2 NP-completeness Results . . . 94
4.2.1 NP-completeness Proof for Perfect Elimination Bipartite Graphs 94
4.2.2 NP-completeness Proof for Star-convex Bipartite Graphs . . . 97
4.2.3 NP-completeness Proof for Dually Chordal Graphs . . . 99
4.3 Hardness Difference . . . 102
4.4 Minimum Maximal Induced Matching in Cographs . . . 107
4.5 Saturated Induced Matching . . . 109

5 Acyclic Matching 111

5.1 Introduction . . . 111
5.2 NP-completeness Results . . . 112
5.2.1 NP-completeness Proof for Star-convex Bipartite Graphs . . . 112
5.2.2 NP-completeness Proof for Comb-convex Bipartite Graphs . . . 114
5.2.3 NP-completeness Proof for Dually Chordal Graphs . . . 116
5.3 Acyclic Matching in Split Graphs . . . 118
5.4 Acyclic Matching in Proper Interval Graphs . . . 121
5.5 Acyclic Matching in Block Graphs . . . 128
5.6 Acyclic Matching in Bipartite Permutation Graphs . . . 133
5.7 Approximation Results . . . 140
5.7.1 Lower Bound on Approximation Ratio . . . 141
5.7.2 APX-completeness for (2k+1)-regular Graphs . . . 145
5.8 Parameterized Complexity . . . 147

6 Minimum Maximal Acyclic Matching 149

6.1 Introduction . . . 149
6.2 NP-completeness Results . . . 150
6.2.1 NP-completeness Proof for Bipartite graphs with maximum degree 6 . . . 150
6.2.2 NP-completeness Proof for Perfect Elimination Bipartite Graphs 155
6.3 Hardness Difference . . . 159

6.4 Minimum Maximal Acyclic Matching in Proper Interval Graphs . . . 165

6.5 Approximation Results 187

6.5.1 Approximation Algorithm 188

6.5.2 Lower Bound on Approximation Ratio 189

6.5.3 APX-completeness for 4-regular Graphs 193

7 Minimum Maximal Uniquely Restricted Matching 197

7.1 Introduction 197

7.2 NP-completeness Results 198

7.2.1 NP-completeness Proof for Chordal Bipartite Graphs 199

7.2.2 NP-completeness Proof for Star-convex Bipartite Graphs . . . 200

7.2.3 NP-completeness Proof for Chordal Graphs 202

7.2.4 NP-completeness Proof for Doubly Chordal Graphs 204

7.3 Complexity Difference 206

7.4 Minimum Maximal Uniquely Restricted Matching in Bipartite Distance-hereditary Graphs 210

7.5 Minimum Maximal Uniquely Restricted Matching in Bipartite Permutation Graphs 212

7.6 Minimum Maximal Uniquely Restricted Matching in Proper Interval Graphs 217

7.7 Minimum Maximal Uniquely Restricted Matching in Threshold Graphs 223

7.8 Approximation Results 224

7.8.1 Approximation Algorithm 224

7.8.2 APX-completeness for Bounded Degree Graphs 225

8 Conclusion and Future Research 229

8.1 Contributions 229

8.2 Open Problems 231

Bibliography 233

Bio-Data **245**

List of Figures

1.1 Graph G. 6

1.2 Graph G_1 admits a $d.i.m.$ $M = \{ab, de, fg\}$ whereas graph G_2 does not admit any $d.i.m.$ 8

1.3 Graph G. 9

1.4 Graph G. 11

1.5 Graph G. 12

1.6 Graph G. 15

1.7 A chain graph. 22

1.8 (a) A bipartite permutation graph G, and (b) the correspondence between the vertices of G and line segments between two parallel lines. 23

1.9 A bipartite permutation graph. 23

1.10 A perfect elimination bipartite graph. 25

1.11 A tree-convex bipartite graph. 25

1.12 A long-k-star-convex bipartite graph. 26

1.13 A circular-convex bipartite graph with its corresponding circular ordering. 26

1.14 A circular-convex bipartite graph, which is not a convex bipartite graph. 27

1.15 The inclusive relationship among different subclasses of bipartite graphs. 28

1.16 A chordal graph. 29

1.17 A doubly chordal graph. 30

1.18 G_1 is an interval graph, which is not a proper interval graph and G_2 is a proper interval graph with its corresponding interval representation. 30

1.19 A block graph. 31

1.20 The inclusive relationship among some classes of graphs studied in this thesis. 32

2.1 An illustration of the construction of H from G. 41

2.2 An illustration of the construction of H from G. 43

2.3 An illustration of the construction of H from G. 44

2.4 A circular-convex bipartite graph. 46

2.5 Algorithm to compute a maximum weight induced matching in a circular-convex bipartite graph. 48

2.6 A triad-convex bipartite graph G with a corresponding triad T. . . . 49

2.7 A triad-convex bipartite graph. 50

2.8 Algorithm to compute a maximum weight induced matching in a triad-convex bipartite graph. 57

3.1 Graphs showing valid partial W-B coloring. 60

3.2 Bipartite graph G_I for $I = \{S_1, S_2, S_3\} = \{(x_1, x_3), (x_2, x_4), (x_2, x_3)\}$. . 63

3.3 Graph showing contradiction in Theorem 3.4.5 (b),(c). 70

3.4 Algorithm to compute a dominating induced matching in a star-convex bipartite graph. 76

3.5 Algorithm to compute a dominating induced matching in a star-convex bipartite graph. 80

3.6 Algorithm to compute a dominating induced matching in a long-k-star-convex bipartite graph. 84

3.7 Algorithm to compute a dominating induced matching in a circular-convex bipartite graph. 90

4.1 An illustration of the construction of G' from G. 95

4.2 An illustration of the construction of $\overline{G}$ from G'. Here, the dashed edges shows a minimum maximal induced matching in G'. 96
4.3 An illustration of the construction of G' from G. 99
4.4 An illustration of the construction of G' from G. 101
4.5 An example of GC_3 graph. 102
4.6 Example showing contradiction in Theorem 4.4.1. 108
4.7 Algorithm to compute a saturated induced matching in a bipartite graph. 110

5.1 An illustration of the construction of H from G. 113
5.2 An illustration of the construction of H from G. 115
5.3 An illustration of the construction of H from G. 117
5.4 Algorithm to compute a maximum acyclic matching in a proper interval graph. 126
5.5 A proper interval graph G. Here the dashed edges represent a maximum acyclic matching obtained by algorithm AM-PIG(G). 127
5.6 A block graph G. 129
5.7 A counterexample to algorithm MaxAM_Cardinality(G). 137
5.8 An illustration of the construction of H from G. 141
5.9 Algorithm to compute an acyclic matching in a graph G. 145

6.1 Gadget F_i for x_i. 151
6.2 The graph G_ψ corresponding to the instance $\psi = (x_1 \vee x_2 \vee \overline{x}_3) \wedge (\overline{x}_1 \vee \overline{x}_2 \vee x_3)$. 152
6.3 An illustration of the construction of G' from G. 156
6.4 An example of GC_4 graph. 160
6.5 Algorithm to compute a maximal acyclic set in a graph G. 161
6.6 An example of GC_3 graph. 162

6.7 Algorithm to compute a minimum maximal acyclic matching in a proper interval graph. 185

6.8 A proper interval graph G^1. The dashed edges represent a minimum maximum acyclic matching obtained by algorithm MMAM-PIG(G^1).187

6.9 Algorithm to compute a maximal acyclic matching in a graph G. . . . 189

6.10 An illustration of the construction of graph $G_{\psi,\alpha}$ corresponding to the instance $\psi = (x_1 \vee \overline{x}_2 \vee x_3) \wedge (\overline{x}_1 \vee x_2 \vee \overline{x}_3)$. An edge from a vertex v to a box represents $p\alpha$ edges between v and each vertex in that box. . 190

6.11 Graph H^t_{ij}. 194

6.12 Subgraph H_{ij} of G' corresponding to edge v_iv_j in G. 194

7.1 An illustration of the construction of H from G. 202

7.2 An illustration of the construction of H from G. 205

7.3 An example of GC_4 graph. 207

7.4 Algorithm to compute a minimum maximal uniquely restricted matching in a bipartite permutation graph. 216

7.5 Algorithm to compute a minimum maximal uniquely restricted matching in a proper interval graph. 222

7.6 Graph $H^t_{ij}, t \in \{1,2\}$. 226

7.7 Subgraph H_{ij} of H corresponding to edge v_iv_j in G. 226

List of Symbols

Symbol	Meaning
$\mathbb{N}$	Set of *natural numbers*
$\forall x$	For *all* x
$x \in X$	x is a *member* of X
$A \subseteq X$	A is a *subset* of X
$\|X\|$	The *cardinality* of a set X
$A \cap B$	The *intersection* of A and B
$A \cup B$	The *union* of A and B
$A \setminus B$	The *set difference* of A and B
$A \times B$	The *Cartesian product* of A and B
$\emptyset$	The *empty* set
$\square$	The *end of a proof*
$\mid$	such that
$E(G)$	The set of *edges* of G
$V(G)$	The set of *vertices* of G
$d(v)$ or $d_G(v)$	The *degree* of the vertex v
Δ or $\Delta(G)$	The *maximum degree* in G
δ or $\delta(G)$	The *minimum degree* in G

$N(v)$ or $N_G(v)$	*neighborhood* of v in G
$N[v]$ or $N_G[v]$	*closed neighborhood* of v in G
$G[H]$	The subgraph of G *induced* by the vertices of H
$G = (X, Y, E)$	The *bipartite graph* G with vertex set $X \cup Y$
$\overline{G}$	The *complement* of an undirected graph G
$\omega(G)$	The *clique number* of G
K_n	The *complete graph* on n vertices
C_n	The *cycle* on n vertices
P_n	The *path* on n vertices
$K_{m,n}$	The *complete bipartite graph* on $m + n$ vertices partitioned into an m-independent set and an n-independent set
$\chi(G)$	The *chromatic number* of G
$\mu(G)$	The *matching number* of G
$\mu'(G)$	The *minimum maximal matching number* of G
$\mu_{in}(G)$	The *induced matching number* of G
$\mu'_{in}(G)$	The *minimum maximal induced matching number* of G
$\mu_{ac}(G)$	The *acyclic matching number* of G
$\mu'_{ac}(G)$	The *minimum maximal acyclic matching number* of G
$\mu_{urm}(G)$	The *uniquely restricted matching number* of G
$\mu'_{urm}(G)$	The *minimum maximal uniquely restricted matching number* of G
P	The class of *deterministic* polynomial-time solvable problems
NP	The class of *nondeterministic* polynomial-time solvable problems
APX	The class of *polynomial-time constant approximable* problems