# Designing Optimized Sparse Arrays for Enhanced Degree of Freedom in Direction of Arrival Estimation

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by

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*Submitted*

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*Dedicated to my beloved mother,*
*Ms. Alka Goel*

# THESIS CERTIFICATE

This is to certify that the thesis titled **Designing Optimized Sparse Arrays for Enhanced Degree of Freedom in Direction of Arrival Estimation**, submitted by **Kretika Goel (2019IDZ8311)**, to the Indian Institute of Technology Delhi, for the award of the degree of **Doctor of Philosophy**, is a bona fide record of the research work done by her under our supervision. The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

monika

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Kretika Goel

# ABSTRACT

Developing a novel category of 2D sparse arrays to improve information resolution with a minimum of possible actual sensors has proven to be a notable and persistently demanding objective. This thesis introduces a new layout for planar sparse arrays to estimate signal arrival angles in both azimuth and elevation dimensions. Specifically, our method involves strategically positioning sensors to maximize the Degree of Freedom(DoF), reduce the occurrence of gaps within the difference coarray, and make it hole-free to enhance its suitability for Direction of Arrival Estimation(DoA) estimation algorithms. To assess the effectiveness of the suggested array, numerical simulations were conducted, demonstrating the achieved Root Mean Square Error(RMSE) value in the proposed Rectangular Coprime Planar Array(RCPA) is the least, i.e., less than 0.1%, compared to other existing planar arrays. Consequently, there is significant interest in devising sparse arrays with sizable difference coarrays and expanding the analysis to encompass additional array characteristics like symmetry, resilience, and cost-effective engineering. We present a scalable and systematic methodology for designing large sparse arrays by introducing sparsity with fractal arrays to create a hole-free difference coarray, which not only increases the number of degrees of freedom in fractal arrays but also aids in enhancing the accuracy of DoA estimation for predicting a maximum number of uncorrelated sources with a minimum possible actual sensors. First, the 1D sparse fractal array is constructed, then extended to a 2D sparse fractal array for both azimuth and elevation angle estimation. Comprehensive robustness analysis was conducted on the proposed sparse fractal array, including one-dimensional (1D) and two-dimensional (2D) configurations, in response to sensor failures by attaining the minimum possible fragility value of 0.5 for $SFA_{1D}$ and 0.6491 for $SFA_{2D}$. Finally, the proposed arrays were rigorously tested for Joint Communication and Sensing (JCAS) applications to evaluate whether they meet the established performance characteristics outlined in the literature, confirming their suitability for effective use in JCAS systems.

# सारांश

न्यूनतम संभावित वास्तविक सेंसर के साथ सूचना संकल्प में सुधार करने के लिए दो आयामी विरल सरणियों की एक नई श्रेणी विकसित करना एक उल्लेखनीय और लगातार मांग वाला उद्देश्य साबित हुआ है। यह थीसिस एजिमुथ और ऊंचाई दोनों आयामों में सिग्नल आगमन कोणों का अनुमान लगाने के लिए प्लानर विरल सरणियों के लिए एक नया लेआउट प्रस्तुत करती है। विशेष रूप से, हमारी विधि में स्वतंत्रता की डिग्री को अधिकतम करने, अंतर सह–सरणी के भीतर अंतराल की घटना को कम करने और आगमन अनुमान एल्गोरिदम की दिशा के लिए इसकी उपयुक्तता को बढ़ाने के लिए इसे छेद–मुक्त बनाने के लिए रणनीतिक रूप से सेंसर की स्थिति बनाना शामिल है। सुझाए गए सरणी की प्रभावशीलता का आकलन करने के लिए, संख्यात्मक सिमुलेशन आयोजित किए गए, जो प्रस्तावित आयताकार सह–अभाज्य प्लानर सरणी में प्राप्त रूट माध्य वर्ग त्रुटि मूल्य को प्रदर्शित करते हैं, जो अन्य मौजूदा प्लानर सरणियों की तुलना में सबसे कम है, यानी 0.1% से कम है। परिणामस्वरूप, बड़े अंतर वाले सह–सरणी के साथ विरल सरणियों को तैयार करने और समरूपता, लचीलापन और लागत प्रभावी इंजीनियरिंग जैसी अतिरिक्त सरणी विशेषताओं को शामिल करने के लिए विश्लेषण का विस्तार करने में महत्वपूर्ण रुचि है। हम एक छेद–मुक्त अंतर सह–सरणी बनाने के लिए फ्रैक्टल सरणियों के साथ विरलता को पेश करके बड़े विरल सरणियों को डिजाइन करने के लिए एक स्केलेबल और व्यवस्थित पद्धति प्रस्तुत करते हैं, जो न केवल फ्रैक्टल सरणियों में स्वतंत्रता की डिग्री की संख्या को बढ़ाता है, बल्कि न्यूनतम संभव वास्तविक सेंसर के साथ अधिकतम संख्या में असंबंधित स्रोतों की भविष्यवाणी करने के लिए आगमन की दिशा के अनुमान की सटीकता को बढ़ाने में भी सहायता करता है। सबसे पहले, एक आयामी विरल फ्रैक्टल सरणी का निर्माण किया जाता है, फिर एजिमुथ और ऊंचाई कोण अनुमान दोनों के लिए दो आयामी विरल फ्रैक्टल सरणी तक बढ़ाया जाता है। प्रस्तावित विरल फ्रैक्टल सरणी पर एक–आयामी और दो–आयामी विन्यासों सहित व्यापक मजबूती विश्लेषण किया गया, जिसमें एक आयामी विरल फ्रैक्टल सरणी के लिए 0.5 और दो आयामी विरल फ्रैक्टल सरणी के लिए 0.6491 का न्यूनतम संभव नाजुकता मूल्य प्राप्त करके सेंसर विफलताओं के जवाब में किया गया। अंत में, प्रस्तावित सरणियों का संयुक्त संचार और संवेदन अनुप्रयोगों के लिए कठोरता से परीक्षण किया गया ताकि यह मूल्यांकन किया जा सके कि क्या वे साहित्य में उल्लिखित स्थापित प्रदर्शन विशेषताओं को पूरा करते हैं, जो जेसीएएस प्रणालियों में प्रभावी उपयोग के लिए उनकी उपयुक्तता की पुष्टि करता है।

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# ABBREVIATIONS

| | |
|---|---|
| **DoF** | Degree of freedom |
| **DoA** | Direction of arrival |
| **CPA** | Coprime Planar Array |
| **GCPA** | Generalized Coprime Planar Array |
| **MRA** | Minimum Redundancy Array |
| **MHA** | Minimum Hole Array |
| **SFA** | Sparse Fractal Array |
| **OBA** | Open Box Array |
| **MUSIC** | Multiple Signal Classification |
| **ESPIRIT** | Estimation of Parameters by Rotational Invariance Techniques |
| **RMSE** | Root Mean Square Error |
| **SNR** | Signal-to-Noise Ratio |
| **URA** | Uniform Rectangular Arrays |
| **NFA** | Nested Fractal Array |
| **CFA** | Coprime Fractal Array |
| **AUGGENIFA** | Augmented GenI Fractal Array |
| **AUGGENIIFA** | Augmented GenII Fractal Array |
| **SNFA** | Super Nested Fractal Array |
| **RNPA** | Rectangular Nested Planar Array |
| **RCPA** | Rectangular Coprime Planar Array |
| **R**$C_d$**PA** | Rectangular CADiS Planar Array |
| **R**$C_s$**PA** | Rectangular CACIS Planar Array |
| **RAG1PA** | Rectangular AUGGEN1 Planar Array |
| **RAG2PA** | Rectangular AUGGEN2 Planar Array |
| **CRLB** | The Cramer-Rao Lower Bound |
| **JCAS** | Joint Communication and Sensing |
| **FNBW** | First Null Beamwidth |
| **HPBW** | Half Power Beamwidth |
| **SLL** | Side Lobe Level |
| **OFDM** | Orthogonal Frequency-Division Multiplexing |
| **FMCW** | Frequency-Modulated Continuous Wave |

# NOTATION

| | |
|---|---|
| $\theta_i$ | Angle of azimuth in degrees |
| $\phi$ | Angle of elevation in degrees |
| $\lambda$ | Wavelength |
| $(\cdot)^T$ | Transpose operation |
| $E[\cdot]$ | Statistical expectation operator |
| $(\cdot)^H$ | Hermitian Transpose |
| $\odot$ | Khatri-Rao product |
| $\otimes$ | Kronecker product |
| $\mathbb{S}$ | Positions of the sensors in Physical Array |
| $\mathbb{D}$ | Difference coarray |
| $\boldsymbol{y(t)}$ | Array observation vector |