NON-STANDARD FINITE ELEMENT METHODS FOR VARIATIONAL INEQUALITIES

RITESH

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY DELHI

JULY 2024

NON-STANDARD FINITE ELEMENT METHODS FOR VARIATIONAL INEQUALITIES

by

Ritesh

Department of Mathematics

Submitted
in fulfillment of the requirements of the degree of
Doctor of Philosophy
to the

Indian Institute of Technology Delhi
July 2024

Dedicated to My Parents

Smt. Neelam Singla

Sh. Ramswaroop Singla

Certificate

This is to certify that the thesis entitled **NON-STANDARD FINITE ELEMENT METHODS FOR VARIATIONAL INEQUALITIES** submitted by **Mr. Ritesh** to the **Indian Institute of Technology Delhi**, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi
July 2024

Prof. Kamana Porwal
Department of Mathematics
Indian Institute of Technology Delhi

Acknowledgments

This thesis marks both the conclusion of one phase and the beginning of a new one in my academic journey. Throughout this transformative journey, my research advisor, colleagues, family, and friends played pivotal roles in fostering my growth and progress. I extend my deepest gratitude to all individuals and institutions who have made this thesis a reality, making my graduate experience an enduring treasure.

First and foremost, I express my deep gratitude and profound appreciation to Dr. Kamana Porwal, my PhD supervisor, whose exceptional mentorship, invaluable insights, and steadfast support have been instrumental throughout my years of research. Dr. Porwal's dedication to the realm of research has consistently inspired me to pursue excellence in mathematics. Her support not only includes providing mathematical guidance for my research but also ensuring adequate funds for its execution. I am grateful for her kindness, support, and encouragement, and the constructive challenges she posed, pushing me to evolve as a better mathematician. Her exceptional guidance has been truly invaluable, and I hold her assistance in the highest regard.

My gratitude also extends to the members of my Student Research Committee (SRC): Prof. Mani Mehra, Prof. Harish Kumar, and Prof. Deepak Patil. Their valuable suggestions, time, and insightful feedback, coupled with constructive criticism, significantly elevated the quality of my research work. I appreciate the time and effort they invested in reviewing my work, keeping me engaged with relevant questions and suggestions that kept me moving forward.

I wish to acknowledge the Indian Institute of Technology, Delhi for providing me with financial support in the form of Research Scholar Travel Award (RSTA). This award facilitated my travel to Germany for the Chemnitz Finite Element Symposium and the ICIAM 2023 conference in

Tokyo, Japan. These opportunities provided an invaluable platform to present my research at an international level, exposing it to a diverse audience.

Heartfelt thanks are extended to the professors at TIFR-CAM Bangalore, particularly Prof. G. D. Veerappa Gowda, Prof. Mythily Ramaswamy, Prof. Sreekar Vadlamani, and Prof. Imran H. Biswas, whose rigorous teaching motivated my pursuit of a research career in partial differential equations. I am grateful to my undergraduate teachers, Prof. Rajni Mendiratta, Prof. Ritu Arora, Prof. Dhanpal Singh, and Prof. Ashish Bansal for their unwavering support and dedicated teaching, which encouraged me to go for higher studies. I also appreciate the support from the system administrators, librarians, and department members at the Department of Mathematics, IIT Delhi, for their academic, technical, and overall assistance.

Special thanks are owed to friends, including Dr. Pratibha Verma, Navnit Yadav, Dr. Lipsy Gupta, Bhawna Dharra, Ravi Kumar Mittal, who stood by me in all my good and bad times, for advising on research, life or sometimes just a place to go and vent. In addition, I am also grateful to my colleagues from IIT Delhi including Dr. Rohit Khandelwal, Dr. Tanvi Wadhawan, Dr. Divay Garg, and Himanshu Rewri, who provided their academic and emotional support, during this entire journey. Additionally, my gratitude extends to Master's and undergraduate friends Mohit, Randeep, Ravi, Navdeep, Vipin, and all my batchmates for their constant motivation, wholehearted support and assistance in several ways. In reality, it's not possible to list the names of everyone who influenced me. I extend my deepest gratitude to each and every one of them.

I express my deepest appreciation to my entire family, especially my parents and my siblings, for their unwavering love, support, and understanding throughout my academic pursuits. Their constant encouragement and belief in me have been my motivation to strive for excellence. I cannot overlook the immense support and understanding I received from my family during this demanding phase of my life. I dedicate this thesis to them with all my heart.

Last but not the least, I extend my thanks to all those unnamed individuals who, in various capacities, contributed to the successful completion of this thesis. Your contributions, regardless of size, are sincerely appreciated. Finally, I thank almighty God who showered grace and blessings on me to complete this thesis successfully.

New Delhi Ritesh

Abstract

The primary objective of this thesis is to explore non-standard finite element methods in the context of both elliptic and parabolic variational inequalities. In the domain of elliptic variational inequalities, our focus centers on two specific contact problems: the obstacle problem and the Signorini problem, serving as prototypes for the elliptic variational inequality of the first kind. Transitioning to parabolic variational inequalities, we conduct a detailed convergence analysis of the finite element method for the fourth-order time dependent obstacle problem. The regularity of the solution plays a pivotal role in determining the qualitative behaviour of the approximation. However, the solution of the linear elliptic problem exhibits singularities owing to changes in boundary conditions, irregular coefficients, and reentrant corners in the domain. This incomplete elliptic regularity of the solution renders uniform refinement ineffective, failing to achieve an optimal convergence rate. Adaptive refinement emerges as a solution to this challenge. This approach relies on *a posteriori* error estimators to enhance efficiency by locally refining the mesh in areas where irregular behaviour in the solution is observed. In this thesis, a theoretical framework is developed to derive these *a posteriori* error estimates and obtain convergence rates for some variational inequalities. The presented theoretical results offer crucial insights for assessing the reliability of computed solutions. Numerical investigations provide empirical support for the theoretical findings presented in this thesis.

This thesis comprises of six chapters, encompassing an introductory chapter and a concluding chapter. The first chapter establishes the groundwork for subsequent sections by conducting a thorough review of relevant literature, consolidating established results, and introducing essential preliminaries. Chapter 2 focuses on deriving pointwise *a posteriori* error estimates for non-conforming finite element method in the context of the second-order elliptic obstacle problem. The analysis heavily relies on standard *a priori* estimates for the regularized Green's function and sub and super solutions corresponding to the continuous solution u. Chapter 3 conducts *a posteriori* analysis of a non-conforming method for the Signorini problem in the supremum norm, which arises in the study of frictionless contact problems. A mapping from the non-conforming finite element space to the conforming finite element space is constructed and extensively used in the analysis. Chapter 4 is dedicated to a detailed development of *a posteriori* error analysis of hybrid higher order method for the obstacle problem in the energy norm. Here, the discrete solution $\underline{u}_h$ is considered as a constant polynomial inside each cell and a linear polynomial on each face. The cornerstone of the method is a local (element-wise) discrete gradient reconstruction operator, and a linear averaging function is employed to transfer the gradient reconstruction operator to its conforming approximation. Moving to Chapter 5, the focus shifts to the numerical solution of a parabolic variational inequality. Spatially semi-discrete and fully-discrete schemes for the fourth-order time dependent obstacle problem are studied using the backward Euler's method in time and the discontinuous Galerkin finite element method in space. To validate the theoretical results, numerical results are presented at the end of each chapter from Chapter 2 to Chapter 5. Finally, in the last chapter, a comprehensive overview of the completed work is presented, alongside suggestions for potential extensions and areas for future investigation.

सार

इस थीसिस का प्राथमिक उद्देश्य अण्डाकार और परवलयिक परिवर्तनशील असमानताओं दोनों के संदर्भ में गैर-मानक परिमित तत्व विधियों का पता लगाना है। अण्डाकार परिवर्तनीय असमानताओं के क्षेत्र में, हमारा ध्यान दो विशिष्ट संपर्क समस्याओं पर केंद्रित है: बाधा समस्या और सिग्नोरिनी समस्या, जो पहले प्रकार की अण्डाकार परिवर्तनीय असमानता के लिए प्रोटोटाइप के रूप में कार्य करती है। परवलयिक परिवर्तनशील असमानताओं में संक्रमण करते हुए, हम चौथे क्रम की समय पर निर्भर बाधा समस्या के लिए परिमित तत्व विधि का विस्तृत अभिसरण विश्लेषण करते हैं। समाधान की नियमितता सन्निकटन के गुणात्मक व्यवहार को निर्धारित करने में महत्वपूर्ण भूमिका निभाती है। हालाँकि, रैखिक अण्डाकार समस्या का समाधान सीमा स्थितियों, अनियमित गुणांक और डोमेन में पुनर्प्रवेश कोनों में परिवर्तन के कारण विलक्षणता प्रदर्शित करता है। समाधान की यह अपूर्ण अण्डाकार नियमितता एकसमान शोधन को अप्रभावी बना देती है, एक इष्टतम अभिसरण दर प्राप्त करने में विफल हो जाती है। अनुकूली परिशोधन इस चुनौती के समाधान के रूप में उभरता है। यह दृष्टिकोण उन क्षेत्रों में जाल को स्थानीय रूप से परिष्कृत करके दक्षता बढ़ाने के लिए पोस्टीरियर त्रुटि अनुमानकों पर निर्भर करता है जहां समाधान में अनियमित व्यवहार देखा जाता है। इस थीसिस में, इन पूर्ववर्ती त्रुटि अनुमानों को प्राप्त करने और कुछ परिवर्तनशील असमानताओं के लिए अभिसरण दर प्राप्त करने के लिए एक सैद्धांतिक ढांचा विकसित किया गया है। प्रस्तुत सैद्धांतिक परिणाम गणना किए गए समाधानों की विश्वसनीयता का आकलन करने के लिए महत्वपूर्ण अंतर्दृष्टि प्रदान करते हैं। संख्यात्मक जांच इस थीसिस में प्रस्तुत सैद्धांतिक निष्कर्षों के लिए अनुभवजन्य समर्थन प्रदान करती है।

इस थीसिस में छह अध्याय शामिल हैं, जिसमें एक परिचयात्मक अध्याय और एक समापन अध्याय शामिल है। पहला अध्याय प्रासंगिक साहित्य की गहन समीक्षा करके, स्थापित परिणामों को समेकित करके और आवश्यक प्रारंभिकताओं को प्रस्तुत करके बाद के अनुभागों के लिए आधार तैयार करता है। अध्याय 2 दूसरे क्रम की अण्डाकार बाधा समस्या के संदर्भ में गैर-अनुरूप परिमित तत्व विधि के लिए बिंदुवार पश्चवर्ती त्रुटि अनुमान प्राप्त करने पर

केंद्रित है। विश्लेषण नियमित रूप से ग्रीन के फ़ंक्शन और निरंतर समाधान यू के अनुरूप उप और सुपर समाधानों के लिए मानक प्राथमिक अनुमानों पर निर्भर करता है। अध्याय 3, सर्वोच्च मानदंड में सिग्नोरिनी समस्या के लिए एक गैर-अनुरूप विधि का एक पश्च विश्लेषण करता है, जो घर्षण रहित संपर्क समस्याओं के अध्ययन में उत्पन्न होता है। गैर-अनुरूप परिमित तत्व स्थान से अनुरूप परिमित तत्व स्थान तक मानचित्रण का निर्माण किया जाता है और विश्लेषण में बड़े पैमाने पर उपयोग किया जाता है। अध्याय 4 ऊर्जा मानक में बाधा समस्या के लिए हाइब्रिड उच्च क्रम विधि के पश्चवर्ती त्रुटि विश्लेषण के विस्तृत विकास के लिए समर्पित है। यहां, असतत समाधान उह को प्रत्येक कोशिका के अंदर एक स्थिर बहुपद और प्रत्येक चेहरे पर एक रैखिक बहुपद के रूप में माना जाता है। विधि की आधारशिला एक स्थानीय (तत्व-वार) असतत ग्रेडिएंट पुनर्निर्माण ऑपरेटर है, और ग्रेडिएंट पुनर्निर्माण ऑपरेटर को उसके अनुरूप सन्निकटन में स्थानांतरित करने के लिए एक रैखिक औसत फ़ंक्शन नियोजित किया जाता है। अध्याय 5 की ओर बढ़ते हुए, ध्यान परवलयिक परिवर्तनशील असमानता के संख्यात्मक समाधान पर केंद्रित हो जाता है। चौथे क्रम की समय पर निर्भर बाधा समस्या के लिए स्थानिक रूप से अर्ध-असतत और पूरी तरह से अलग योजनाओं का अध्ययन समय में पिछड़े यूलर की विधि और अंतरिक्ष में असंतत गैलेरकिन परिमित तत्व विधि का उपयोग करके किया जाता है। सैद्धांतिक परिणामों को प्रमाणित करने के लिए, अध्याय 2 से अध्याय 5 तक प्रत्येक अध्याय के अंत में संख्यात्मक परिणाम प्रस्तुत किए जाते हैं। अंत में, अंतिम अध्याय में, भविष्य की जांच के लिए संभावित विस्तार और क्षेत्रों के सुझावों के साथ, पूर्ण किए गए कार्य का एक व्यापक अवलोकन प्रस्तुत किया जाता है।

Contents

List of Figures

List of Symbols

Symbol	**Meaning**
$\mathcal{T}_h$	a regular triangulation of Ω,
T	a simplex of $\mathcal{T}_h$,
$\lvert T\rvert$	volume of the simplex T,
h_T	diameter of the simplex T,
h	$\max\{h_T : T \in \mathcal{T}_h\}$,
$h_{\min}$	$\min\{h_T : T \in \mathcal{T}_h\}$,
Γ	boundary of Ω,
Γ_D	Dirichlet boundary,
Γ_N	Neumann boundary,
Γ_C	potential contact boundary,
F	a face of $\mathcal{T}_h$,
$\mathrm{mid}(F)$	midpoint/barrycenter of the edge/face F,
h_F	diameter of the edge/face F,
$\mathcal{F}_h^i$	set of all interior edges/faces of $\mathcal{T}_h$,
$\mathcal{F}_h^b$	set of all edges/faces of $\mathcal{T}_h$ that are on $\partial\Omega$,
$\mathcal{F}_h^D$	set of all Dirichlet boundary edges/faces,
$\mathcal{F}_h^C$	set of all contact boundary edges/faces,
$\mathcal{F}_h^N$	set of all Neumann boundary edges/faces,
$\mathcal{F}_h$	set of all edges/faces of $\mathcal{T}_h$,
$\mathcal{F}_T$	set of all edges/faces of the simplex T,
p	a node of a simplex T,
$\mathcal{F}_p$	set of all edges/faces connected to a given node p,
$\mathcal{T}_h(p)$	$\{T \in \mathcal{T}_h \colon p \cap T \neq \emptyset\}$,

$\lvert\mathcal{T}_h(p)\rvert$	cardinality of $\mathcal{T}_h(p)$,
$\mathcal{V}_h$	set of all vertices in $\mathcal{T}_h$,
$\mathcal{V}_T$	set of vertices of the simplex T,
$\mathcal{V}_h^i$	set of all vertices in $\mathcal{T}_h$ that are in Ω,
$\mathcal{V}_h^b$	set of all vertices of $\mathcal{T}_h$ that are on $\partial\Omega$,
$\mathcal{V}_h^D$	set of all vertices on Γ_D,
$\mathcal{V}_h^N$	set of all vertices on Γ_N,
$\mathcal{V}_h^C$	set of all vertices on Γ_C,
$\mathbb{P}^k(T)$	the space of polynomials of degree atmost k over element T,
$\mathbb{P}^k(\mathcal{T}_h)$	$\{v \in L^2(\Omega)\colon v\vert_T \in \mathbb{P}^k(T)\ \forall\ T \in \mathcal{T}_h\}$,
π_T^k	L^2-orthogonal projection onto $\mathbb{P}^k(T),\ T \in \mathcal{T}_h$,
$\mathbb{P}^k(F)$	the space of polynomials of degree atmost k over face F,
$\mathbf{n}_{TF}$	unit normal vector pointing outward from T from face $F \in \mathcal{F}_T$,
π_F^k	L^2-orthogonal projection onto $\mathbb{P}^k(F),\ F \in \mathcal{F}_h$,
ω_p	$\bigcup\limits_{T\in\mathcal{T}_h(p)} T$, patch around node p,
ω_T	$\bigcup\limits_{T'\cap T\neq\emptyset} T'$,
ω_F	$\bigcup\limits_{F\subset\partial T} T$, for any face F,
∇_h	piecewise (element-wise) gradient,
$L^p(\Omega)$	Lebesgue spaces, $1 \le p \le \infty$,
$W^{m,p}(\Omega)$	Sobolev spaces, $1 \le p \le \infty$,
$H^m(\Omega),\ H_0^m(\Omega)$	Sobolev spaces $W^{m,p}(\Omega)$ for $p = 2$,
$H^m(\Omega, \mathcal{T}_h)$	piecewise Sobolev space of order $m \in \mathbb{N}$,
$\lvert\lvert\cdot\rvert\rvert_{m,p,D}$	$\lvert\lvert\cdot\rvert\rvert_{W^{m,p}(D)}$ where $m \in \mathbb{Z}\setminus\{0\}$ and $1 \le p \le \infty$ and $D \subseteq \Omega$,
$\lvert\lvert\cdot\rvert\rvert_{m,D}$	$\lvert\lvert\cdot\rvert\rvert_{H^m(D)}$ where $m \in \mathbb{Z}\setminus\{0\}$ and $D \subseteq \Omega$,
$\lvert\cdot\rvert_{m,p,D}$	the associated semi norm on $W^{m,p}(D)$ where $D \subseteq \Omega$,
$\lvert\cdot\rvert_{m,D}$	the associated semi norm on $H^m(D)$ where $D \subseteq \Omega$,
$\lvert\lvert\cdot\rvert\rvert_{\infty,D}$	$\lvert\lvert\cdot\rvert\rvert_{L^\infty(D)}$, where $D \subseteq \Omega$,
$\lVert\cdot\rVert$	the associated norm on $L^2(\Omega)$,
$\lVert\cdot\rVert_{L^2(D)}$	the associated norm on $L^2(D)$ where $D \subsetneq \Omega$,
$(\cdot,\cdot)$	the associated inner product on $L^2(\Omega)$,
$(\cdot,\cdot)_D$	the associated inner product on $L^2(D)$ where $D \subsetneq \Omega$,
$\delta_{i,j}$	the Kronecker's Delta i.e., $\delta_{i,j} = 1$ if $i = j$ and 0 elsewhere,
$\boldsymbol{v}'$	the transpose of vector valued function $\boldsymbol{v}$,

$C^m(\Omega)$	the space of m times continuously differentiable functions on $\Omega,\ m \geq 0$, integer,
$\mathcal{D}(\Omega)$	the space of infinitely differentiable functions having compact support in Ω,
C	generic positive constant, independent of perturbation and mesh parameters,
$X \lesssim Y$	$X \leq CY$ for some positive constant C, which is independent of the mesh size h,
$C_c^m(\Omega)$	the space of m times continuously differentiable functions with compact support in Ω,
$meas(A)$	measure of the set $A \subseteq \bar{\Omega}$,
$supp(f)$	support of the function f.