THE COFINALLY COMPLETE METRIC SPACES AND THEIR RELATIVES

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THE COFINALLY COMPLETE METRIC SPACES AND THEIR RELATIVES

by

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Dedicated to

My Parents

Certificate

This is to certify that the thesis entitled **The Cofinally Complete Metric Spaces and their Relatives** submitted by **Miss Manisha Aggarwal** to the Indian Institute of Technology Delhi, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bona fide research work carried out by her under my guidance and supervision. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi
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Abstract

This thesis studies different families of complete metric spaces, namely cofinally complete spaces, Atsuji spaces, Bourbaki-complete spaces and cofinally Bourbaki-complete spaces, in an extensive manner. Each of these spaces is characterized by the clustering of certain sequences which are generalizations of Cauchy sequences. More precisely, the cofinally complete spaces, the Bourbaki-complete spaces and the cofinally Bourbaki-complete spaces are those in which every cofinally Cauchy sequence, Bourbaki-Cauchy sequence and cofinally Bourbaki-Cauchy sequence clusters respectively, whereas the Atsuji spaces are characterized by the clustering of pseudo-Cauchy sequences of distinct terms.

In this thesis, we give a comprehensive list of equivalent characterizations of these spaces. Since all these spaces are complete, we also concentrate on characterizations of the metric spaces whose completions are one of the aforesaid spaces. The main emphasis is on the equivalent conditions based on certain sequences and functions. Thus, various functions similar to Cauchy-continuous functions are defined which preserve the aforesaid sequences. Interestingly, we establish the relations of these functions with continuous functions, Cauchy-continuous functions and some Lipschitz-type functions and thereby nice characterizations of the complete metric spaces are obtained. In order to get more familiar with those classes of functions, their various properties are also discussed along with their comparison with each other and consequently many examples are given for illustration. In fact, their sequential characterizations are investigated as well and for that reason some sequences are defined which are similar to asymptotic and uniformly asymptotic sequences. Subsequently, those metric spaces are characterized on which such functions are bounded and as a consequence we obtain many equivalent conditions for total boundedness and finite chainability.

Contents

List of Symbols

Symbol	Meaning
$\forall$	for all
$\exists$	there exists
$\in$	belongs to
$\notin$	does not belong to
$\subseteq$	subset or equal
$\cup$, $\cap$	union, intersection
$X \setminus E$ or E^c	the complement of E in X
$\emptyset$	empty set
$\mathbb{N}$	the set of natural numbers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the real line
l_2	the Hilbert space of square summable real sequences
(e_n)	the standard orthonormal basis in l_2
$\square$	end of a proof

$\lvert\cdot\rvert$	the usual distance metric on $\mathbb{R}$
$\lVert\cdot\rVert$	norm
$(x_n)_{n\in\mathbb{N}}$	a sequence in a non-empty set, occasionally it may be denoted by (x_n)
$f\vert_A$	the restriction of f to A where $f: X \to Y$ is a function, $\emptyset \neq A \subseteq X$

For the following notations, (X, d) is a metric space.

$\overline{A}$ or $cl_X A$	the closure of A in X
$intA$ or A°	the interior of A in X
X'	the set of accumulation points in X
$(\widehat{X}, d)$	the completion of a metric space (X, d)
$B_d(x,\delta)$ or $B(x,\delta)$	the open ball in (X, d), centered at $x \in X$ with radius $\delta > 0$
$C_d(x,\delta)$ or $C(x,\delta)$	the closed ball in (X, d), centered at $x \in X$ with radius $\delta > 0$