A THESIS ON
HIGHER ORDER METHODS FOR THE SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

By

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C E R T I F I C A T E

This is to certify that the thesis entitled "Higher Order Methods for the Solution of Ordinary Differential Equations" which is being submitted by Miss Saroj Sachdev for the award of Doctor of Philosophy (Mathematics) to the Indian Institute of Technology, Delhi, is a record of bonafide research work. She has worked for the last three and half years under my guidance and supervision.

The thesis has reached the standard fulfilling the requirements of the regulations relating to the degree. The results obtained in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

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CONTENTS

S Y N O P S I S

This thesis represents some numerical methods for the numerical integration of the initial value problems in ordinary differential equations. The complete discussion of the thesis has been carried out in four chapters of which the first one gives some basic definitions and exhibits the current stage of the step by step numerical methods used to integrate the ordinary differential equations. The rest of the thesis develops some higher order repeated quadrature formulae using Fehlberg's transformation, for integrating initial-value problems in linear ordinary differential equations of various orders. A few more ideas developed here show some properties of implicit Runge-Kutta methods for first order and second order non-linear differential equations using a suitable transformation. A brief synopsis of the contents of the thesis is as follows:

CHAPTER I

This chapter gives an exposition of the current stage of the step by step methods available for the numerical integration of ordinary differential equations. This involves the comparitive study of the existing methods and suggests some new methods for the integration of ordinary differential equations which have been developed in the following chapters.

CHAPTER II

In order to approach our problem gradually, we derive some higher order repeated quadrature formulae to evaluate any higher order integral directly. As we have seen that an n-fold integral

$$\int_{x_0}^{x_0+h} \int_{x_0}^{x} \ldots\ldots \int_{x_0}^{x} f(x)\ (dx)^n \ , \ (n \geqslant 2)$$

is usually reduced to a single integral and then its numerical value is find out by some numerical quadrature formulae. Using Fehlberg's transformation, new higher order repeated quadrature formulae has been used to evaluate the above integral which compares quite favourably with Fillipi's method. In the former case not only the accuracy gets increased by (n-1), but the truncation error also gets reduced considerably. To illustrate the main feature of the method an example has been computed and the results have been compared with the Fillipi's method and the exact solution.

In the latter half of this chapter, we consider the application of the repeated quadrature formulae, to the numerical solution of the linear second order differential equation.

$$Y''(x) = f(x)Y(x)+g(x) \ , \ Y(x_0)=Y_0 \ ; \ Y'(x_0)=Y_0'.$$

The numerical solution of this differential equation has been obtained by first transforming the differential equation by Fehlberg's transformation which makes the first (m+n) - total derivative of the transformed dependent variable vanish at the initial point (where n being the order of the ordinary differential equation) and thereby reduces the double integration in the solution, by a sum of evaluations of functions by the repeated quadrature formulae. The method has been extended to the linear equation

$$Y^{(n)}(x) = f(x)\ Y(x) + g(x)$$

with initial conditions

$$Y^{(v)}(x_0) = Y_0^v \ , \ v = 0,1,\ldots\ldots\ldots, n-1.$$

Ordinarily the numerical integration of higher order differential equation is carried out by solving an equivalent system of first order differential equations. The main feature in this chapter consists in proving that the accuracy achieved by solving the n-th order differential equations directly can be increased by (n-1) provided we make use of the repeated quadrature formulae as developed in the earlier part of the chapter. Following the later criterian, the numerical solution of the second order differential equation has also been obtained by Fillipi's method. A stability criterian is also discussed for the second order differential equation. Sample examples of the differential equations of various orders have been computed to illustrate the comparison of this method with the exact solution.

CHAPTER III

In this chapter we have derived implicit Runge-Kutta methods of order (m+5) and (m+6), for the first order non-linear differential equation by transforming the dependent variable suitably and by making use of the evaluations of the m-total derivatives at the initial point. This transformation is so chosen that it makes the first (m+n)-total derivatives of the transformed dependent variable vanish at the initial point and also $\frac{\partial f}{\partial Y^{(r)}}$ vanishes at the initial point for $r=0,1,\ldots, n-1$, where n is the order of the differential equation, Y is the transformed dependent variable and f is the function in the transformed differential equation. This behaviour of the derivatives reduce considerably the amount of labour in developing the formulas of higher order. Formulas have been computed for m = 0(8)1. The stability region has also been obtained for every value of m, for the formulas of order (m+5) and (m+6). A few numerical examples have been computed using the implicit Runge-Kutta formulas of order h^{m+6} and the approximate solution has been compared with the exact solution.

CHAPTER IV

This chapter extends the methods of chapter III to the second order non-linear differential equation. Using Lotkin's bound on the derivatives, the implicit Runge-Kutta formulae of order (m+5) and (m+6) have been developed. Stability region has also been determined for the formulas based upon 2-stage and 3-stage implicit Runge-Kutta formulae, which has been iterated further to achieve the desired accuracy. A numerical example has been computed which shows a good agreement of the computed value with the exact solution.