A THESIS

ON

NUMERICAL METHODS FOR ORDINARY DIFFERENTIAL EQUATIONS

By

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CERTIFICATE

This is to certify that the thesis entitled, "Numerical Methods For Ordinary Differential Equations", which is being submitted by Shri John S.V. Saldanha for the award of the degree, Doctor of Philosophy (Mathematics), to the Indian Institute of Technology, Delhi, is a bonafide record of research work. He has worked for the last three years and five months under my guidance and supervision.

The thesis has reached the standard fulfilling the requirements of the regulations relating to the degree. The results obtained in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

M.K. Jain 22/6/76

(Professor M.K. JAIN)

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SYNOPSIS

Almost all scientific and technological investigations result in mathematical models, of which systems of ordinary differential equations is one, which occurs quite frequently. The need to obtain accurate approximations to the solutions of these sets of equations has engaged the attention of mathematicians throughout and has produced a number of methods. The advent of the digital computer has greatly aided the numerical analyst in his search for and analysis of new and efficient techniques.

This thesis presents some methods for the numerical integration of ordinary differential equations and is divided into four chapters, of which the first one is devoted to a discussion of stiff systems and derivation of a stiffly stable linear multi-step method, making use of second derivatives. In the second and third chapters methods of numerical integration are used for the solution of second order two point boundary value problems. This quadrature approach is extended in the fourth chapter to the solution of a fourth order linear two point boundary value problem occurring in plate deflection theory.

The contents of the thesis briefly summarized are as follows:

<u>Chapter I</u>:

In Chapter I a discussion is made of the concept of stability of linear multi-step methods and stiff stability in particular. It is known that for the efficient integration of systems of equations whose jacobian matrix has widely separated eigenvalues, A-stable methods are the best. But the accuracy of A-stable linear multi-step methods is severely restricted, since their order cannot exceed two. To get over this difficulty, Gear defined the concept of stiff stability which enabled him to obtain stiffly stable methods of order upto six,

Based on this definition, stiffly stable formulae of the type

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\beta_k y'_{n+k} + h^2\gamma_k y''_{n+k} , \quad k = 2(1)10$$

are developed in the second part of this chapter, making use of second derivatives. They are shown to have order k+1 and their convergence proved. The stability curves of these methods are drawn and the results of the numerical experiments on three examples compared with the actual solutions.

Chapter II:

In the first part of this chapter integration rules are developed for evaluating $\int_{-1}^{1}(1-x)f(x)dx$ with weight functions corresponding to Radau, Lobatto and Gauss-Legendre integration. These rules are used for obtaining difference schemes for the solution of the two point boundary value problem,

$$y''(x) = F(x,y), \quad x \in [a,b], \; y(a) = \alpha , \; y(b) = \beta .$$

Six different methods, each of order six, are presented which depend on only function evaluations unlike the derivative based formula of Usmani. The adaptability of these schemes to non-linear equations is also shown. Two of the methods depend on only one off-step point thereby contributing to a saving of computer time. It is significant to note that all the methods give better results than h^4-extrapolation of the well known Numerov scheme. In the case of the linear equation, least truncation error is found to occur in the application of the method based on Gauss quadrature. Further, the round-off level is reached at $h = 2^{-7}$ which is an improvement on the result reported by Usmani. These conclusions are based on the results of the numerical experiments done.

The methods developed here are shown to be useful for the solution of the characteristic value problem

$$y''(x) + Ay = 0, \quad y(0) = y(1) = 0.$$

The characteristic values can be determined to a higher order of accuracy by these formulae.

Chapter III:

Five of the difference schemes developed in Chapter II are used in Chapter III for the solution of the mixed boundary value problem

$$y''(x) = F(x,y) \quad , \quad x \in [a,b],$$
$$y'(a) - cy(a) = \alpha,$$
$$y'(b) + dy(b) = \beta .$$

The quadrature rules are again found useful for evaluating the integrals in the identities

$$y(x_1) \equiv y(x_o) + hy'(x_o) + h^2\int_0^1 (1-t)y''(x_o + ht)dt$$

and

$$y(x_{N-1}) \equiv y(x_N) - hy'(x_N) + h^2\int_0^1 (1-t)y''(x_N - ht)dt.$$

All the five schemes are shown to be of order six and their convergence proved.

Numerical experiments show that the five methods give almost the same order of accuracy and hence it is difficult to make a choice of any one method as the best. Round-off level in this case is reached at $h = 2^{-8}$.

<u>Chapter IV</u>:

The quadrature approach for the solution of boundary value problems shown in Chapters II and III is extended in Chapter IV to the solution of the fourth order linear two point boundary value problem

$$y^{iv}(x) + f(x)y(x) = g(x), \quad f(x) \geq 0, \quad x \in [a,b].$$

As in Chapter II, quadrature rules are first established for evaluating

$$\int_{-1}^{1} (1-x)^s f(x)dx$$

with weight functions corresponding to Radau and Lobatto integration. It is found that all of them lead to one and the same difference scheme,

$$\delta^4 y_n = \frac{h^4}{6}[y_{n-1}^{iv} + 6y_n^{iv} + y_{n+1}^{iv}], \quad n = 2(1)N-2.$$

Monotone nature of the coefficient matrix is established and hence the convergence of the method. Two examples are solved to show the usefulness of the method in the solution of the differential equation occurring in plate deflection theory.

A sixth order scheme is also proposed, though its convergence is not proved. From the results of the numerical experiments, it is conjectured that the method must be convergent.

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