

SOME RESULTS IN WEIGHTED ENUMERATION AND ELLIPTIC IDENTITIES

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SOME RESULTS IN WEIGHTED ENUMERATION AND ELLIPTIC IDENTITIES

by

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To
My Mother and My Husband
Sunita and Sukhdarshan

Certificate

This is to certify that the thesis entitled **Some Results in Weighted Enumeration and Elliptic Identities** submitted by **Ms. Archana Kumari** to the Indian Institute of Technology Delhi, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bona fide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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Abstract

This thesis studies the elliptic analogues of identities from combinatorics and hypergeometric series. Specifically, we extend the Fibonacci numbers, several other elementary identities, and summation and transformation formulas from hypergeometric series. We extend Fibonacci numbers with arbitrary weights and generalize a dozen Fibonacci identities. We propose an elliptic extension different from that given by Schlosser and Yoo. The proofs of most of the identities are combinatorial, extending the proofs given by Benjamin and Quinn, and in the q -case, by Garrett. Some identities are proved by telescoping. We provide elliptic extensions of elementary identities such as the sum of the first n odd or even numbers, the geometric sum, and the sum of the first n cubes. In the course of our study, we obtained an identity with many parameters, which appears to be new even in the q -case. In addition, we recover some q -identities due to Warnaar. We provide an alternate approach to obtaining expansion formulas on the lines of the well-poised Bailey lemma. We recover results due to Spiridonov and Warnaar and one new formula of this type. As a result, we are able to derive 19 new transformation formulas for elliptic hypergeometric series. Among these transformation formulas, eight appear to be new even in the basic hypergeometric case, when $p = 0$.

Abstract

यह शोधप्रबंध संयोजनिकी और अतिज्यामितीय श्रेणियों की समानताओं के दीर्घवृत्त समरूपों का अध्ययन करता है। विशेष रूप से, हम फिबोनाच्ची संख्याओं, कुछ अन्य प्रारंभिक गणितीय समानताओं तथा अतिज्यामितीय श्रेणियों के योग और रूपांतरण सूत्रों का विस्तार करते हैं। हम फिबोनाच्ची संख्याओं का विस्तार मनमाने भारों के साथ करते हैं और लगभग एक दर्जन फिबोनाच्ची समानताओं का सामान्यीकरण प्रस्तुत करते हैं। हम एक ऐसा दीर्घवृत्त विस्तार प्रस्तुत करते हैं जो पूर्ववर्ती शोधकर्ताओं द्वारा प्रदत्त विस्तार से भिन्न है। अधिकांश समानताओं के प्रमाण संयोजनात्मक विधियों द्वारा दिए गए हैं, जो बेंजामिन और किन द्वारा प्रस्तुत प्रमाणों का विस्तार करते हैं, और q -समरूप के सन्दर्भ में गैरेट के कार्य पर आधारित हैं। कुछ समानताओं को संकोचक विधि द्वारा सिद्ध किया गया है। हम प्रारंभिक गणितीय समानताओं जैसे पहले n विषम या सम संख्याओं का योग, गुणोत्तर श्रेणी का योग, तथा पहले n घनों का योग के दीर्घवृत्त समरूप प्रस्तुत करते हैं। अध्ययन के दौरान, हमें अनेक चरयुक्त एक समानता प्राप्त हुई है, जो q -समरूप के रूप में भी नवीन प्रतीत होती है। इसके अतिरिक्त, हम वॉर्नर द्वारा दी गई कुछ q -समरूप समानताओं को पुनः प्राप्त करते हैं। हम एक वैकल्पिक पद्धति प्रस्तुत करते हैं, जिसके माध्यम से संतुलित विस्तार सूत्र प्राप्त होते हैं। इस प्रक्रिया में हम स्पिरिडोनोव और वॉर्नर के कुछ परिणामों के साथ-साथ इस प्रकार की एक नवीन समानता भी प्राप्त करते हैं। इस प्रकार, हम दीर्घवृत्त अतिज्यामितीय श्रेणियों के लिए कुल 19 नवीन रूपांतरण सूत्र सिद्ध करते हैं। इन सूत्रों में से आठ सूत्र मूलभूत अतिज्यामितीय अवस्था (जब $p = 0$ हो) में भी नवीन प्रतीत होते हैं।

Contents

Certificate	i
Acknowledgements	iii
Abstract	v
1 Introduction	1
1.1 Elliptic functions	6
1.2 The modified theta functions	7
1.3 An elliptic number	8
1.4 Elliptic hypergeometric series	11
1.5 Telescoping	15
2 Elliptic Extension of Elementary Identities	19
2.1 Introduction	19
2.1.1 Some important specializations of the elliptic numbers and elliptic weights	20
2.2 Elementary Examples	21
2.2.1 Special cases of (2.2.1)	22
2.2.2 Special cases of (2.2.6)	24
2.2.3 Some special cases of (2.2.12)	26
2.3 An esoteric identity and some special cases	27
2.3.1 Some special cases of (2.3.1)	31

3	Elliptic Extension of Fibonacci Numbers	33
3.1	Introduction	33
3.2	Weighted Fibonacci numbers	35
3.2.1	Special cases	36
3.3	Fibonacci numbers from weighted counting	37
3.4	Proofs that really weigh	40
3.5	Fibonacci identities from telescoping	45
4	Elliptic Hypergeometric Series	49
4.1	Introduction	49
4.2	Expansion formulas	53
4.2.1	First expansion formula	54
4.2.2	Second expansion formula	55
4.2.3	Third expansion formula	56
4.2.4	Fourth expansion formula	57
4.3	Elliptic transformation formulas	58
4.3.1	Transformation formulas obtained from the first expansion formula	59
4.3.2	Transformation formulas obtained from the second expansion for- mula	61
4.3.3	Transformation formulas from the third expansion formula	64
4.3.4	Transformation formulas obtained from the fourth expansion . . .	66
4.4	Remarks on q -hypergeometric special cases	70
4.5	Special cases of elliptic hypergeometric series	72
4.5.1	Some special cases of (4.5.2)	74
4.6	Concluding remarks	76
	Appendix I: SAGEMATH Code for Verification of Results	77
	References	85
	Bio-Data	91