EXPLICATIVE & NUMERICAL SOLUTIONS OF SOME QUEUEING MODELS

by

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Dedicated to

My parents, my wife and children

CERTIFICATE

This is to certify that the thesis entitled "**Explicative & Numerical Solutions of Some Queueing Models**" being submitted by **Mr. Ahmad Mohamed Kamel Tarabia** for the award of the degree of the **DOCTOR OF PHILOSOPHY** in **MATHEMATICS** is a bonafide record of research work done under my guidance and supervision.

The thesis has reached the standard fulfilling the requirements of the regulations relating to the degree. The results obtained in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

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A.M.K. Tarabia

April, 1998 AHMED M. K. TARABIA

Abstract

Queueing models have proven to be a versatile tool for modelling and evaluating complex systems in various fields such as operation research (flexible manufacturing systems, repair systems, traffic networks), computer science (computing systems, program behaviour), telecommunication research and biology (population models, infection models). Successful modelling in all these areas depends heavily on the possibility of explicitly determining the behaviour of the underlying Markov processes which are introduced to describe the model's evolution over time. Mostly researchers have investigated the queueing models by considering the probability generating functions or Laplace transform technique and applying it on the differential-difference equations. However, these techniques fail if we have a large set of differential-difference equations which are too cumbersome to handle even for a steady-state solution. Our attempt is to find an alternative approach to analyse some queueing models and to obtain even the transient probabilities without using the classical technique such as the probability generating function or any of the known numerical method.

The approach is based on expressing the transient probability as infinite series whose coefficients satisfy simple recurrence relations which in turn allow the rapid and efficient evaluation of the probabilities.

The present study investigates the single server and multi-server queues. Using the proposed technique we obtain a new closed form for the transient probabilities of the single server Markovian queue as well as the busy period distribution for the multi-server queue. We also use the proposed approach to other models of queueing networks such as the queue in series (tandem queues) and shortest queue (jockeying in queues). The efficacy of the results derived are illustrated by numerical computations. Moreover, a brief comparison of our approach and other methods is given with regard to the CPU time for computations.

The thesis is divided into six chapters. Each chapter gives a brief introduction about the problem considered and presents development of the theory and finally, its applications and numerical computations are also shown. The details of the different chapters are given below.

CHAPTER - I

This chapter includes a broad survey of the earlier related work which motivated the study of the problems investigated in this thesis. It also contains and explains some fundamental concepts of queueing theory emphasizing the aspects relevant to this thesis. Some important examples of networks such as, queues in series and shortest queue are presented.

CHAPTER - II

This chapter discusses the single server Markovian queue with finite and infinite waiting space. Here, we have come a long way from the available results of $M/M/1/N$ queue in the literature, which are so complex that it is almost impossible to obtain the results for the infinite case $M/M/1/\infty$ by making $N \longrightarrow \infty$ in those results. Here, we succeed in obtaining a simpler closed form formula for the transient state probabilities for $M/M/1/N$ queue which can be expressed as:

$$\begin{aligned} p_n(t) &= e^{-(\lambda+\mu)t} \rho^n \sum_{m=0}^{\infty} \left\{ \sum_{r=0}^{\left[\frac{m-n}{2}\right]} \sum_{i=0}^{\left[\frac{r}{N-1}\right]} A(m, r-(N+1)i)\rho^r \right. \\ &\qquad \left. - \sum_{r=\left[\frac{m-n}{2}\right]+1}^{m-N} \sum_{i=0}^{\left[\frac{m-N-r}{N-1}\right]} A(m, r+(N+1)(i+1))\rho^r \right\} \frac{(\mu t)^m}{m!}, \\ &\qquad\qquad 0 \le n \le N,\ \rho \ne 1 \\ &= e^{-2\mu t} \sum_{m=0}^{\infty} \left\{ \sum_{i=0}^{S_1} \begin{pmatrix} m \\ \left[\frac{m-n}{2}\right] - (N+1)i \end{pmatrix} \right. \end{aligned}$$

$$+\sum_{i=0}^{S_2}\binom{m}{\left[\frac{m-n}{2}\right]+(N+1)(i+1)}\Bigg)\Bigg\}\frac{(\mu t)^m}{m!},\quad 0\leq n\leq N,\quad \rho=1$$

where

$$A(m,s)=\begin{cases}\binom{m}{s}-\binom{m}{s-1} & ,\quad s>0\\ 0 & ,\quad s<0 \text{ or } s>m+1\end{cases}$$

with

$$S_1=\left[\frac{\left[\frac{m-n}{2}\right]}{N+1}\right]\quad,\quad S_2=\left[\left(m-N-1-\left[\frac{m-n}{2}\right]\right)/(N+1)\right]$$

and $p_n(t)$, λ, μ & $\rho=\lambda/\mu$ are the usual symbols used for this model. The infinite case results come out directly by taking the limit $N\longrightarrow\infty$, we obtain

$$p_n(t) \;=\; e^{-(1+\rho)\mu t}\rho^n\sum_{m=0}^{\infty}\left\{\sum_{r=0}^{\left[\frac{m-n}{2}\right]}\left\{\binom{m}{r}-\binom{m}{r-1}\right\}\rho^r\right\}\frac{(\mu t)^m}{m!},\quad \rho\neq 1$$

so that as $\rho\longrightarrow 1$ and $N\longrightarrow\infty$ we get

$$p_n(t) \;=\; e^{-2\mu t}\sum_{m=0}^{\infty}\binom{m}{\left[\frac{m-n}{2}\right]}\frac{(\mu t)^m}{m!},\quad \rho=1$$

which agree with the results as reported in Sharma & Bunday (1997), for $M/M/1/\infty$ queue. By making $t\longrightarrow\infty$ we also obtain easily the well known steady state result for $\rho<1$. It is also shown that the mean and variance of queue length can be easily obtained numerically from the transient state probabilities. Comparison in CPU time required by the new formula and other solutions of htis model is presented. Recently Conolly & Langaris (1993) have shown the equivalence of the classical Bessel functions form and Sharma & Shobha's series formula (1984) for the transient state probabilities of $M/M/1/\infty$ queue. In this chapter, using simple algebra we also establish that all the three results concerning $M/M/1/\infty$ queue namely classical Bessel functions form, Sharma and Shobha (1984) formula and Sharma & Bunday (1997) formula are all equivalent.

CHAPTER - III

The concept of busy period was introduced by Borel in 1942 and since then it has almost become an integral part of the study of any queueing system. This chapter analyses the multichannel Markovian queue using the proposed approach in this thesis in Chapter 2. A closed form expression is obtained for probability density function of the duration of a busy period which can be expressed as

$$b(t) = c\mu e^{-(\lambda+c\mu)t} \sum_{m=0}^{\infty} \left\{ \binom{m}{\left[\frac{m}{2}\right]} - \binom{m}{\left[\frac{m-1}{2}\right]} \right\} \rho^{\left[\frac{m}{2}\right]} \frac{(c\mu t)^m}{m!}, \quad \rho = \lambda / c\mu$$

whence its moments of any desired order can be explicitly obtained with the help of the following theorem

Theorem 1 *For any non-negative integral k and real $x \neq 1$,*

$$\sum_{m=0}^{\infty} \left\{ \binom{m}{\left[\frac{m}{2}\right]} - \binom{m}{\left[\frac{m-1}{2}\right]} \right\} \binom{m+k}{m} \left(\frac{x}{(1+x)^2} \right)^{\left[\frac{m}{2}\right]} =$$

$$\begin{cases} 1+x & , \quad k=0 \\ \frac{(1+x)^2}{1-x} & , \quad k=1 \\ \frac{(1+x)^{k+1}}{(1-x)^{2k-1}} \sum_{j=0}^{k-2} \binom{k-1}{j} \binom{k-2}{j} \frac{x^j}{j+1} & , \quad k \geq 2 \end{cases}$$

Using this theorem we get the k^{th} order moment as

$$E(T_c^k) = \begin{cases} \frac{1}{c\mu(1-\rho)} & , \quad k=1 \\ \frac{k!}{(c\mu)^k (1-\rho)^{2k-1}} \sum_{j=0}^{k-2} \binom{k-1}{j} \binom{k-2}{j} \frac{\rho^j}{j+1} & , \quad k \geq 2 \end{cases}$$

In the end numerical applications of the cumulative distribution are also highlighted.

CHAPTER - IV

This chapter deals with the transient behaviour of $M/M/c/N$ queueing system. Similar series method introduced in the previous chapters has been extended to this model which enables us to compute the transient solution avoiding eigenvalues computation and requires less CPU time. The distribution of busy period is also computed for different values of t, c and N by obtaining the distribution function.

$$\begin{aligned} B(t) &= \int_0^t b(u)du \\ &= \rho^{c-i} \sum_{m=0}^{\infty} a(m,c) \frac{(c\mu)^{m+1}}{m!} I_m(t) \end{aligned}$$

where $I_m(t)$ satisfies the recurrence relation

$$I_m(t) = -\frac{t^m}{\lambda + c\mu} e^{-(\lambda+c\mu)t} + \frac{m}{\lambda + c\mu} I_{m-1}(t), \quad m \geq 1$$

with

$$I_0(t) = \frac{1}{\lambda + c\mu} \left(1 - e^{-(\lambda+c\mu)t}\right)$$

Moreover, the effect of the traffic intensity on the model's paramaters such as idle probability and average queue length is shown. The effect of the initial condition and the waiting room capacity N on the transient state probabilities and the model's parameters is worked out.

CHAPTER - V

In this chapter, numerical studies on the effect of server arrangements on the transient behaviour of queues in series network are presented. A finite space queueing model consisting of r queues in series with single server facilities at each queue is discussed. Poisson input and exponential service times have been assumed. The computation of the time dependent state probabilities $p(n_1, n_2, \cdots, n_r, t)$ for queues in series is developed by writing the transient probability in the form:

$$\begin{aligned} p(\underline{n}, t) &= e^{-\beta t} \sum_{m=0}^{\infty} a(m, \underline{n}) \frac{(\beta t)^m}{m!}, \\ \underline{n} &= (n_1, n_2, \cdots, n_r) \text{ and } \beta = \lambda + \sum_{i=1}^{r} \mu_i. \end{aligned}$$

Furthermore, numerical results concerning the important characteristics of the system such as average queue length (number of customers in the system), idle probability, overflow probability, utilization and effect of utilization factor on these parameters are carried out. To establish that the sum of the probabilities is one, an analytic proof of the following Lemma is given.

Lemma 1 *For any positive integers m and N (N is fixed)*

$$\sum_{n_1=0}^{N} \sum_{n_2=0}^{N-n_1} \cdots \sum_{n_r=0}^{N-(n_1+n_2+\cdots+n_{r-1})} a(m, n_1, n_2, \cdots, n_r) = 1.$$

CHAPTER - VI

This chapter is devoted to discussion of a queueing network consisting of two queues in parallel being fed by a single stream of Poisson arrivals with rate λ. The service times in each of the two queues are independent and exponentially distributed with parameters μ_1 and μ_2 respectively. The capacity of each queue is finite, say N, i.e., the total number of units the system can take is $2N$ including the ones being served. The arriving customer joins the shortest of the two queues and if these are of equal lengths, he joins either of the queue with probability 1/2. The moment a server becomes idle and there are customers waiting in the other queue, the customer immediately following the customer who is receiving service at that counter is transferred to the idle server's queue. This model is called shortest queue model with jockeying. The first part of this chapter deals with the derivation of the differential-difference equations satisfying the transient probabilities $p_{ij}(t)$. Using the matrix method and the fact that $p_{ij}'(t) = 0$, as

$t \longrightarrow \infty$ the equilibrium equations can be rewritten as

$$\begin{bmatrix} A_1 & B_2 & C_3 & 0 & \cdots & \cdot\cdot & 0 \\ 0 & A_2 & B_3 & C_4 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & \cdot\cdot & 0 & A_{N-2} & B_{N-1} & C_N \\ 0 & \cdots & \cdot\cdot & 0 & 0 & A_{N-1} & B_N \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ \cdot \\ \vdots \\ P_N \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \cdot \\ \vdots \\ 0 \end{bmatrix}$$

where

$$A_k = (a_{ij})_{(2k+1)\times(2k-1)}, \quad k = 1, 2, \cdots, N-1$$

with $a_{2k-1\ 2k-1} = a_{2k\ 2k-1} = \rho$ and $a_{ij} = 0$, otherwise.

$$B_k = (b_{ij})_{(2k-1)\times(2k-1)}, \quad k = 2, 3, \cdots, N$$

with

$$\begin{aligned} b_{ii} &= -2(1+\rho) &&, \quad i = 1, 2, \cdots, 2k-2 \\ b_{i+2\ i} &= 2\rho &&, \quad i = 1, 2, \cdots, 2k-4 \\ b_{2i-1\ 2i+1} &= 2\alpha_1 &&, \\ b_{2i\ 2i+2} &= 2\alpha_2 &&, \quad i = 1, 2, \cdots, k-2 \\ b_{2k-3\ 2k-1} &= 2\alpha_1 &&, \\ b_{2k-2\ 2k-1} &= 2\alpha_2 &&, \quad k = 2, 3, \cdots, N \\ b_{2k-1\ 2k-2} = b_{2k-1\ 2k-3} &= \rho &&, \quad k = 2, 3, \cdots, N \\ b_{2k-1\ 2k-1} &= -(1+\rho) &&, \quad k \neq N \\ b_{2N-1\ 2N-1} = -1 \text{ and } b_{ij} &= 0 &&, \quad \text{otherwise.} \\ C_k &= (C_{ij})_{(2k-3)\times(2k-1)} &&, \quad k = 3, 4, \cdots, N \end{aligned}$$

with

$$\begin{aligned} c_{11} = c_{22} = 2 \quad &, \\ c_{2i-1\,2i-1} = 2\alpha_2 \quad &, \quad c_{2i\,2i} = 2\alpha_1, i = 2,3,\cdots,k-2 \\ c_{2k-3\,2k-2} = \alpha_1 \quad &, \quad c_{2k-3\,2k-3} = \alpha_2, \;\; k = 3,4,\cdots,N \\ \text{and } c_{ij} = 0 \quad &, \quad \text{otherwise.} \end{aligned}$$

which enables us to compute the steady-state solution using the following theorem:

Theorem 2 *For any integer i, $i = 1,2,3,\cdots,N$ we have*

$$\begin{aligned} P_i &= (-1)^{i+1}\frac{\lambda^2}{2\mu_1\mu_2}Q_iA_{i-1}Q_{i-1}A_{i-2}\cdots Q_2A_1p_{00}, \;\; i = 1,2,3,\cdots,N \\ p_{01} &= \frac{\lambda}{2\mu_2}p_{00}, \\ p_{10} &= \frac{\lambda}{2\mu_1}p_{00}, \end{aligned}$$

with

$$p_{00} = \begin{cases} \frac{2\mu_1\mu_2(1-\rho)}{2\mu_1\mu_2(1-\rho)+(\mu_1+\mu_2)^2\rho(1-\rho)+\lambda^2(1-\rho^{2N-1})} & , \;\; \rho \neq 1 \\ \frac{2\mu_1\mu_2}{2\mu_1\mu_2+2N(\mu_1+\mu_2)^2} & , \;\; \rho = 1 \end{cases}$$

and $Q_1 = A_0 = 1$, $Q_N = B_N^{-1}$ *and* $Q_i = \left[B_i - C_{i+1}Q_{i+1}A_i\right]^{-1}$, $i = N-1, N-2, \cdots, 2$ *and* $P_k = \left[p_{1k} \;\; p_{k1} \;\; p_{2k} \;\; p_{k2} \;\; \cdots p_{k-1\,k} \;\; p_{k\,k-1} \;\; p_{kk}\right]^t$, $k = 1,2,\cdots,N.$

The second part of this chapter deals with the computation of the transient probabilities using the proposed method in this thesis by writing the transient state probability $p_{ij}(t)$ as

$$p_{ij}(\tau) = e^{-(1+\rho)\tau}\rho^{i+j}\sum_{m=0}^{\infty} a(m,i,j)\frac{\tau^m}{m!},$$

where $\tau = (\mu_1 + \mu_2)t$ and $\rho = \lambda/(\mu_1 + \mu_2)$.

This transforms the differential - difference equations into difference equations that enables us to compute $a(m,i,j)$ easily and hence $p_{ij}(t)$. Certain other important results have also been worked out. Moreover, to prove $\sum_{i=0}^{N}\sum_{j=0}^{N} p_{ij}(t) = 1$, the following lemmas have been established.

Lemma 2 *For any non-negative integers m and N, $N > 1$, we have*

$$\begin{aligned}
&\sum_{n=1}^{N-1} a(m,1,n)\rho^{n+1} + \sum_{n=2}^{N-1} a(m,n,1)\rho^{n+1} + \\
&\sum_{n=4}^{N-1}\sum_{k=2}^{n-2} a(m,k,n)\rho^{n+k} + \sum_{n=4}^{N-1}\sum_{k=2}^{n-2} a(m,n,k)\rho^{n+k} \\
&+ \sum_{n=3}^{N-1} a(m,n-1,n)\rho^{2n-1} + \sum_{n=3}^{N-1} \rho^{2n-1} a(m,n,n-1) \\
&+ \sum_{n=2}^{N-1} a(m,n,n)\rho^{2n} + \sum_{n=1}^{N} a(m,n,N)\rho^{N+n} \\
&+ \sum_{n=1}^{N-1} a(m,N,n)\rho^{N+n} = \sum_{n=1}^{N}\sum_{k=1}^{N} a(m,n,k)\rho^{n+k}
\end{aligned}$$

Lemma 3 *For non-negative integers m, N, $N > 1$, we have*

$$a(m,0,0) + \rho\left(a(m,0,1) + a(m,1,0)\right) + \sum_{n=1}^{N}\sum_{k=1}^{N} a(m,n,k)\rho^{n+k} = (1+\rho)^m,$$

$$\rho = \lambda/(\mu_1 + \mu_2).$$

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