FROBENIUS-SCHUR INDICATOR AND GEL'FAND CHARACTER

by

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to the

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Dedicated to

My Family

Certificate

This is to certify that the thesis entitled **"Frobenius-Schur Indicator and Gel'fand Character"** submitted by **"Mr. Sunil Kumar Prajapati"** to the Indian Institute of Technology Delhi, for the award of the Degree of Doctor of Philosophy, is a record of the original bona fide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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Abstract

This thesis is concerned with the problems on word equations in finite groups and a question raised by Prof. K.W. Johnson about the total character of a finite group. Let G be a finite group and let $\mathrm{Irr}(G)$ denotes the set of all irreducible characters of G. Suppose f is a map from a non-empty finite set X to G. Define the map $\zeta_G^f : G \longrightarrow \mathbb{N} \cup \{0\}$ given by $g \mapsto |f^{-1}(g)|$. The map $\zeta_G^f(g)$ is the number of solution in G for the equation $f(x) = g$. Suppose $w = w(t_1, t_2, \ldots, t_n)$ is a word in n symbols $t_1, t_2, \ldots, t_n$. Then $w(t_1, t_2, \ldots, t_n)$ defines the map $w : G^n \longrightarrow G$ given by $(g_1, g_2, \ldots, g_n) \mapsto w(g_1, g_2, \ldots, g_n)$. In general, ζ_G^f or ζ_G^w need not be a character of G. We prefer to write ζ_G^k instead of ζ_G^w if $n = 1$ and $w(x) = x^k$.

Bryant and Kovács [Bull. Aust. Math. Soc. 5(2) (1971) 265-269] proved that ζ_G^k is a generalized character of G. We study the following question: "Given a group G, under what conditions ζ_G^k is a character of G?" For a Frobenius group G with the Frobenius kernel N and a Frobenius complement H, we show that if for each $k \in \mathbb{N}$, ζ_N^k and ζ_H^k are characters, then so is ζ_G^k. For a p-group, we show that if G is p-abelian, then ζ_G^k is a character for each $k \in \mathbb{N}$. We also discuss the map ζ_G^k for dihedral groups, generalized quaternion groups, quasidihedral groups, semidihedral groups, and exceptional p-groups.

Strunkov [Izv. Ross. Akad. Nauk Ser. Mat., 59(6) (1995) 171-180] initiated the study of behavior of the map ζ_G^f and proved the following: given $f_i : X_i \longrightarrow G$

$(i = 1, 2)$, if $\zeta_G^{f_1}$ and $\zeta_G^{f_2}$ are characters (resp. generalized characters), then so is ζ_G^f, where $f : X_1 \times X_2 \longrightarrow G$ is given by $f(x_1, x_2) = f_1(x_1)f_2(x_2)$. We generalize this result in various ways and use our results to show that the solution function ζ_G^w for the word equation $w(t_1, t_2, \ldots, t_n) = g$ $(g \in G)$ is a character, where $w(t_1, t_2, \ldots, t_n)$ denotes the product of $t_1, t_2, \ldots, t_n, t_1^{-1}, t_2^{-1}, \ldots, t_n^{-1}$ in a randomly chosen order.

The Gel'fand character of a group G is the sum of all the non-isomorphic irreducible characters of G. It is also known as the total character of G and is denoted by τ_G. Prof. K. W. Johnson raised the following question for a finite group G: "Does there exist an irreducible character χ of G and a monic polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(\chi) = \tau_G$?" It is not difficult to see the answer is negative in general. We call a polynomial $f(x) \in \mathbb{Q}[x]$ a *Johnson polynomial* of G if there exist $\chi \in \mathrm{Irr}(G)$ such that $f(\chi) = \tau_G$. The Johnson's problem has been studied for dihedral groups D_{2n} by Poimenidou and Wolfe [Int. J. Math. Sci., 38 (2003) 2447-2453]. In this thesis, we investigate various groups for existence of a Johnson polynomial. We have proved that if $(G, Z(G))$ is a generalized Camina pair, then a necessary and sufficient condition for G to have a Johnson polynomial is that $Z(G)$ is cyclic. Suppose G is a non-abelian p-group of order p^3 or p^4. Then G has a Johnson polynomial if and only if $Z(G)$ is cyclic and $G' \subseteq Z(G)$. We have also investigated a Johnson polynomial for semidihedral groups and generalized quaternion groups.

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